Existence of Optimal Controls for Functional-Differential Systems on Semi Axis

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We study functional-differential equations on the semi-axis which are nonlinear with respect to the phase variables and linear with respect to the control. Sufficient conditions for existence of optimal control in terms of the right-hand side and the quality criterion are obtained. Connection between the solutions of the problems on infinite and finite intervals is studied and results about these connections are proven.

Let $h>0$ be a constant, describing the delay. By $|\cdot|$ we denote a vector norm in R^d, and by $\|\cdot\|$ the norm of $d\times m$-matrices, which agrees with the vector norm. We introduce the necessary functional spaces which we use in this paper. Let $C=C([-h,0];R^d)$ be the Banach space of continuous functions from $[-h,0]$ into R^d with the uniform norm $\|\varphi\|_C=\max\limits_{\theta\in[-h,0]}|\varphi(\theta)|$, and let $L_p=L_p([-h,0];R^m)$, $p>1$ be the Banach space of p-integrable m-dimensional vector-valued functions with the norm

$$\|\varphi\|_{L_p}=\bigg(\int\limits_{-h}^{0}|\varphi(s)|^p\,ds\bigg)^{1/p}.$$

Let x be continuous function on $[0,\infty)$ and let $\varphi\in C$. If $x(0)=\varphi(0)$, then the function

$$x(t,\varphi)=\begin{cases}\varphi(t), & t\in[-h,0],\\ x(t), & t\geq 0\end{cases}$$

is continuous for $t\geq 0$. In the standard way for each $t\geq 0$ we can introduce an element $x_t(\varphi)\in C$ by the expression $x_t(\varphi)=x(t+\theta,\varphi)$, $\theta\in[-h,0]$. Further, instead of $x_t(\varphi)$ we write x_t.

Let $t\in[0,\infty)$, and D be a domain in $[-h,\infty)\times C$ with boundary ∂D.

In this paper, we study optimal control problems for systems of functional-differential equations ($\dot{x}=dx(t)/dt$)

$$\dot{x}(t)=f_1(t,x_t)+\int\limits_{-h}^{0}f_2(t,x_t,y)u(t,y)\,dy,\ \ t\in[0,\tau],\ \ x(t)=\varphi_0(t),\ \ t\in[-h,0], \tag{1}$$

with one of the next cost criterion

$$J[u]=\int\limits_{0}^{\tau}\big(e^{-\gamma t}A(t,x_t)+B(t,u(t,\cdot\,))\big)\,dt\longrightarrow\inf, \tag{2}$$

$$J[u]=\int\limits_{0}^{\tau}\bigg(e^{-\gamma t}A(t,x_t)+\int\limits_{-h}^{0}|u(t,y)|^2\,dy\bigg)\longrightarrow\inf. \tag{3}$$

These problems are considered on the infinite horison $t \geq 0$. Here $\varphi_0 \in C$ is a fixed element such that $(0, \varphi_0) \in D$, $x(t)$ is the phase vector in R^d, and x_t is the corresponding phase vector in C, τ is the moment when (t, x_t) reaches the boundary ∂D for the first time or $\tau = \infty$ otherwise. Also, $f_1 : D \to R^d$, $f_2 : D \times [-h, 0] \to M^{d\times m}$ – $d \times m$-dimensional matrix such that for each $(t, \varphi) \in D$ $f_2(t, \varphi, \cdot\,)$ belongs to the space $L_q([-h, 0]; M^{d\times m})$ with the norm

$$\|f_2(t,\varphi)\|_{L_q} = \bigg(\int\limits_{-h}^{0} \|f_2(t,\varphi,y)\|^q \, dy \bigg)^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

$A : D \to R^+$, $B : [0, \infty) \times L_p \to R^+$ are given mappings.

The control parameter $u \in L_p([0, \infty) \times [-h, 0])$ is m-dimensional vector function such that for almost all $(t, y), u(t, y) \in W$, $0 \in W$, where W is a convex and closed set in R^m.

For each control function, we define corresponding solution (trajectory) of (1). A continuous function $x(t)$ is a solution of (1) on the interval $[-h, T]$, if it satisfies the following conditions: $x(t) = \varphi_0(t)$, $t \in [-h, 0]$; $(t, x_t) \in D$ for $t \in [0, T]$; for $t \in [0, T]$ $x(t)$ satisfies the integral equation

$$x(t) = \varphi_0(0) + \int\limits_0^t \bigg[f_1(s, x_s) + \int\limits_{-h}^{0} f_2(s, x_s, y) u(s, y) \, dy \bigg] \, ds.$$

The control function $u(t, \cdot\,)$ is considered admissible for the problems (1), (2) and (1), (3), if: $u(t, y) \in L_p([0, \infty) \times [-h, 0]$; $u(t, y) \in W$ for almost all $t \geq 0$, $y \in [-h, 0]$; the solution $x(t)$ corresponding to the control $u(t, \cdot\,)$ exists on the interval $[-h, \tau]$, $\tau > 0$; $|J[u]| < \infty$.

Let $V(\varphi_0)$ denote the Bellman function for the problem on the infinite horison and let $V_T(\varphi_0)$ be the Bellman function for the corresponding problem on some finite interval $[0, T]$.

In [4] it was shown that system (1) includes as particular cases the usual optimal control problem for functional-differential equations

$$\dot{x}(t) = f(t, x_t) + g(t, x_t) u(t), \quad u \in L_p([0, \infty); R^m), \tag{4}$$

for equations with maximum, and for system of ordinary differential equations.

The choice of the control $u(t, \cdot\,) \in L_p([0, \infty); [-h, 0])$ for each t as an element of the function space is justified (determined) by two reasons:

1) the given problem to be similar to the general functional-operator form of an optimal control problem where $u(t) \in W$ and W is a topological space (see, for example, [1]).

2) the given class of problems includes some problems with applications to economics (see [2,3]).

The goal of this work is to generalize the results obtained in [4] to the infinite horison $[0, \infty)$ and to clarify the relation between problems on finite and infinite intervals. It turns out that by the means of optimal control for finite interval, it is possible to construct easily minimizers for the problem on infinite horison.

Let D be a domain in $[-h, \infty) \times C$, and ∂D be its boundary. We introduce the notations $D_t = \{\varphi \in C, (t, \varphi) \in D\}$, $D_c = \bigcup\limits_{t\geq 0} D_t$, where D_c is bounded in C.

Assumption 1. *The admissible controls are m-dimensional vector functions $u(t, y) \in L_p([0, \infty) \times [-h, 0]; R^m)$ such that for almost all $t \geq 0$ and $y \in [-h, 0]$ we have $u(t, y) \in W$, where W is a convex closed set in R^m and $0 \in W$ and there exists $J[u]$.*

The set of admissible controls we denote as $\mathcal{U}$.

Assumption 2. *The mappings $f_1(t,\varphi): D \to R^d$ and $f_2(t,\varphi,y): D\times[-h,0] \to M^{d\times m}$ are defined and measurable with respect to all arguments in the domain D and $D_1 = \{(t,\varphi)\in D, y\in[-h,0]\}$, respectively. Moreover, these functions satisfy in D and D_1, with respect to φ the condition for linear growth and the Lipchitz condition, i.e., there exists constant $K>0$ such that*

$$|f_1(t,\varphi)| + \|f_2(t,\varphi,y)\| \le K(1+\|\varphi\|_C), \tag{5}$$

for $(t,\varphi)\in D$, $y\in[-h,0]$,

$$\big|f_1(t,\varphi_1) - f_1(t,\varphi_2)\big| + \big\|f_2(t,\varphi_1,y) - f_2(t,\varphi_2,y)\big\| \le K\|\varphi_1-\varphi_2\|_C, \tag{6}$$

for $(t,\varphi_1),(t,\varphi_2)\in D$.

Assumption 3.

1) *The mapping $A: D\to R$, $A(t,\varphi)\ge 0$ for $(t,\varphi)\in D$ is defined and continuous in D and for $(t,\varphi)\in D$ there is a constant $K_A>0$ such that $A(t,\varphi)\le K_A(1+\|\varphi\|_C)$;*

2) *the mapping $B:[0,\infty)\times L_p\to R$ is measurable with respect to all its arguments and there are constants $a>0$, $a_1>0$ such that $a_1\|z\|^p_{L_p}\ge B(t,z)\ge a\|z\|^p_{L_p}$ if $t\ge 0$;*

3) *for each $t\ge 0$, $B(t,z)$ is strongly differentiable with respect to z and for $t\ge 0$ and $z\in L_p$ the Frechet derivative $\frac{\partial B}{\partial z}$ satisfies the estimate*

$$\Big\|\frac{\partial B}{\partial z}\Big\|_{\mathcal{L}(L_p;R^1)} \le a_2\|z\|^{p-1}_{L_p}$$

for some constant $a_2>0$, independently of t and z. Here $\|\cdot\|_{\mathcal{L}(L_p;R^1)}$ is the uniform operator norm in the space of linear continuous functionals over L_p.

The main results of this work are given by the following theorems.

Theorem 1. *Suppose that Assumptions* 1–3 *are satisfied. Then there exists a solution (x^*,u^*) of the problems* $(1),(2)$ *and* $(1),(3)$.

Let $T>0$ be fixed. By (x^*_T,u^*_T) we denote the solution of the problems $(1),(2)$ or $(1),(3)$ on $[0,T]$.

For the problem on infinite horison, we define

$$u_{T,\infty}(t,\cdot\,) = \begin{cases} u^*_T(t,\cdot\,), & t\in[0,T],\\ 0, & t>T \end{cases} \tag{7}$$

and $x^{T,\infty}(t)$ is corresponding trajectory.

It is obvious that the given control is admissible for the original problem. Again, $(u^*(t,\cdot\,),x^*(t))$ is an optimal pair for the problem $(1),(2)$, τ – the time at which the solution x^*_t reaches the boundary ∂D.

Theorem 2. *Suppose that Assumptions* 1–3 *are satisfied, then we have:*

1)
$$V_T(\varphi_0)\to V(\varphi_0),\ \ T\to\infty;$$

2) *there is a sequence $T_n\to\infty$, $n\to\infty$, such that the sequence $\{u_{T_n,\infty}\}$ is minimizer for the problem* $(1),(2)$, *i.e.*

$$J[u_{T_n,\infty}] \longrightarrow V,\ \ n\to\infty; \tag{8}$$

3) *there is a sequence $T_n \to \infty$, $n \to \infty$, such that*

$$u_{T_n,\infty} \xrightarrow{w} u^*, \quad n \to \infty \tag{9}$$

weekly in $L_p([0,\infty) \times [-h,0]; R^m)$;

4) *pointwise on $[0,\tau^*]$, uniformly on each finite interval*

$$x^{T_n,\infty}(t) \longrightarrow x^*(t), \quad n \to \infty.$$

If the problem $(1),(2)$ *has unique solution, then the convergence in* (8), (9) *occurs for all $T \to \infty$.*

Remark. In the conditions of Theorem 2 for the functional (3) all statements of Theorem 2 are valid, if the weak convergence of optimal controls (9) is replaced with strong convergence in $L_2([0,\infty) \times [-h,0]; R^m)$.

The next theorem is about the case when the domain D_c in the statement of the problem is unbounded. As it is shown in [4], the solution of the original problem cannot go to infinity in finite time. However, it can increase without bound in such a way that the integrals in (2) and (6) become divergent for all admissible controls. Now we give a theorem which guarantees existence of optimal control in this case. So, we assume that it is possible that D is unbounded domain in $[-h,\infty) \times C$ but the set of control values W is bounded in R^m. Without loss of generality, we can assume that W is a ball with radius r.

Theorem 3. *If the conditions of Theorem* 1 *are satisfied and $\gamma < (hr+1)K$, then the problems* $(1),(2)$ *and* $(1),(3)$ *have solutions.*

References

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