On the Solvability of the Boundary value Problem for One Class of Higher-Order Semilinear Partial Differential Equations

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In the cylindrical domain $D_T := \Omega \times (0,T)$, where Ω is a Lipschitz domain in $\mathbb{R}^n$, consider a boundary value problem on finding a solution $u = u(x,t)$ to the equation

$$L_f := \frac{\partial^{4k} u}{\partial t^{4k}} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\Big(a_{ij}\frac{\partial u}{\partial x_i}\Big) + f(u) = F \tag{1}$$

by the boundary conditions

$$u\big|_{\Gamma} = 0, \tag{2}$$

$$\frac{\partial^i u}{\partial t^i}\bigg|_{\Omega_0 \cup \Omega_T} = 0, \ \ i = 0, \dots, 2k-1, \tag{3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given continuous function, $a_{ij} = a_{ji} = a_{ij}(x) \in C^1(\overline{\Omega})$, $i,j = 1,\dots,n$, $F = F(x,t)$ are the given, and $u = u(x,t)$ is an unknown real functions, k is a natural number, $n \geq 2$. Here $\Gamma := \partial\Omega \times (0,T)$ is the lateral face of the cylinder D_T, $\Omega_0 : \ x \in \Omega, \ t = 0$ and $\Omega_T : \ x \in \Omega, \ t = T$ are upper and lower bases of this cylinder, respectively.

Below, we assume that operator $K := \sum\limits_{i,j=1}^{n} \frac{\partial}{\partial x_j}\big(a_{ij}(x)\frac{\partial u}{\partial x_i}\big)$ is evenly elliptical in $\overline{\Omega}$, i.e.

$$k_0|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq k_1|\xi|^2 \ \ \forall\, x \in \overline{\Omega}, \ \ \xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n, \tag{4}$$

where $k_0, k_1 = const > 0$, $|\xi|^2 = \sum\limits_{i=1}^{n} \xi_i^2$. Note that (4) implies the hypoellipticity of the linear part of operator L_f from (1), i.e. L_0 is hypoelliptic for each $x = x_0 \in \overline{\Omega}$.

Denote by $C^{2,4k}(\overline{D}_T, \partial D_T)$ the space of functions u continuous in $\overline{D}_T$, having continuous partial derivatives $\frac{\partial u}{\partial x_i}$, $\frac{\partial^2 u}{\partial x_i \partial x_j}$, $\frac{\partial^l u}{\partial t^l}$, $i,j = 1,\dots,n$; $l = 1,\dots,4k$, in $\overline{D}_T$. Assume

$$C_0^{2,4k}(\overline{D}_T, \partial D_T) := \bigg\{ u \in C^{2,4k}(\overline{D}_T) : \ u\big|_{\Gamma} = 0, \ \frac{\partial^i u}{\partial t^i}\bigg|_{\Omega_0 \cup \Omega_T} = 0, \ i = 0,\dots,2k-1 \bigg\}.$$

Introduce the Hilbert space $W_0^{1,2k}(D_T)$ as a completion with respect to the norm

$$\|u\|^2_{W_0^{1,2k}(D_T)} = \int\limits_{D_T} \bigg[u^2 + \sum_{i=1}^{2k} \Big(\frac{\partial^i u}{\partial t^i}\Big)^2 + \sum_{i=1}^{n} \Big(\frac{\partial u}{\partial x_i}\Big)^2 \bigg]\, dx\, dt$$

of the classical space $C_0^{2,4k}(\overline{D}_T, \partial D_T)$.

Remark 1. From definition of the space $W_0^{1,2k}(D_T)$ it follows that if $u \in W_0^{1,2k}(D_T)$, then $u \in \overset{\circ}{W}{}_2^1(D_T)$ and $\frac{\partial^i u}{\partial t^i} \in L_2(D_T)$, $i = 2, \dots, 2k$. Here $W_2^1(D_T)$ is the well-known Sobolev space consisting of the elements of $L_2(D_T)$, having the first order generalized derivatives from $L_2(D_T)$, and $\overset{\circ}{W}{}_2^1(D_T) = \big\{u \in W_2^1(D_T) : \ u|_{\partial D_T} = 0\big\}$, where the equality $u|_{\partial D_T} = 0$ is understood in the sense of the trace theory.

Below, on the function $f = f(u)$ we impose the following requirements

$$f \in C(\mathbb{R}), \ \ |f(u)| \le M_1 + M_2|u|^\alpha, \ \ u \in \mathbb{R}, \tag{5}$$

where $M_i = const \ge 0$, $i = 1, 2$, and

$$0 \le \alpha = const < \frac{n+1}{n-1}\,. \tag{6}$$

Remark 2. The embedding operator $I : W_2^1(\overline{D}_T) \to L_q(D_T)$ represents a linear continuous compact operator for $1 < q < \frac{2(n+1)}{n-1}$, when $n > 1$. At the same time the Nemitski operator $N : L_q(D_T) \to L_2(D_T)$, acting by the formula $Nu = -f(u)$, due to (5) is continuous and bounded if $q \ge 2\alpha$. Thus, since due to (6) we have $2\alpha < \frac{2(n+1)}{n-1}$, then there exists a number q such that $1 < q < \frac{2(n+1)}{n-1}$ and $q \ge 2\alpha$. Therefore, in this case the operator

$$N_0 = NI : \overset{\circ}{W}{}_2^1(D_T) \to L_2(D_T)$$

will be continuous and compact. Besides, from $u \in W_0^{1,2k}(D_T)$ it follows that $f(u) \in L_2(D_T)$ and, if $u_m \to u$ in the space $W_0^{1,2k}(D_T)$, then $f(u_m) \to f(u)$ in the space $L_2(D_T)$.

Definition 1. Let function f satisfy the conditions (5) and (6), $F \in L_2(D_T)$. The function $u \in W_0^{1,2k}(D_T)$ is said to be a weak generalized solution of the problem (1)–(3), if for any $\varphi \in W_0^{1,2k}(D_T)$ the integral equality

$$\begin{aligned}\int\limits_{D_T} \bigg[\frac{\partial^{2k} u}{\partial t^{2k}} \cdot \frac{\partial^{2k}\varphi}{\partial t^{2k}} + \sum_{i,j=1}^n a_{ij}(x)\,\frac{\partial u}{\partial x_i}\,\frac{\partial \varphi}{\partial x_j}\bigg]\,dx\,dt + \int\limits_{D_T} f(u)\varphi\,dx\,dt \\ = \int\limits_{D_T} F\varphi\,dx\,dt \ \ \forall\,\varphi \in C_0^{2,4k}(\overline{D}_T, \partial D_T)\end{aligned}$$

is valid.

It is not difficult to verify that if the solution of the problem (1)–(3) in the sense of Definition 1 belongs to the class $C_0^{2,4k}(D_T, \partial D_T)$, then it will also be a classical solution of this problem.

Theorem. *Let the conditions* (5), (6) *and*

$$\lim_{|u|\to\infty} \inf \frac{f(u)}{u} \ge 0 \tag{7}$$

be fulfilled. Then for any $F \in L_2(D_T)$ the problem (1)–(3) *has at least one weak generalized solution $u \in W_0^{1,2k}(D_T)$.*

Remark 3. Let us note that if along with the conditions (5)–(7) imposed on function f to demand that it is monotonous, then the solution $u \in W_0^{1,2k}(D_T)$ of the problem (1)–(3), the existence of which is stated in the theorem, is unique. As show the examples, when the conditions imposed on nonlinear function f are violated, then the problem (1)–(3) may not have a solution.