# Relationships Between Different Kinds of Stochastic Stability for Functional Differential Equations

**Ramazan I. Kadiev**

*Dagestan Research Center of the Russian Academy of Sciences & Department of Mathematics, Dagestan State University, Makhachkala, Russia*

*E-mail:* kadiev_r@mail.ru

**Arcady Ponosov**

*Department of Mathematical Sciences and Technology, Norwegian University of Life Sciences, P. O. Box 5003, N-1432 Ås, Norway*

*E-mail:* arkadi@nmbu.no

## 1 Notation and preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a stochastic basis (see, e.g. [5]), where $\Omega$ is a set of elementary probability events, $\mathcal{F}$ is a $\sigma$-algebra of all events on $\Omega$, $(\mathcal{F}_t)_{t\geq 0}$ is a right continuous family of $\sigma$-subalgebras of $\mathcal{F}$, $P$ is a probability measure on $\mathcal{F}$; all the above $\sigma$-algebras are assumed to be complete with respect to (w.r.t. in what follows) the measure $P$, i.e. they contain all subsets of zero measure; the symbol $E$ stands for the expectation related to the probability measure $P$.

In the sequel, we use an arbitrary yet fixed norm $|\cdot|$ in $R^n$, the real-valued index $p$ satisfying the assumption $0 \leq p \leq \infty$ and a continuous positive function $\gamma(t)$ defined for all $t \geq 0$.

By $Z = (z_1, \dots, z_m)^T$ we denote an $m$-dimensional semimartingale (see, e.g. [5]). A most popular particular case of $Z$ is the standard Brownian motion (the Wiener process) $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_m)^T$.

The general linear stochastic functional differential equation is defined as follows (see, e.g. [2]):

$$dx(t) = (Vx)(t)\,dZ(t) \ \ (t \geq 0), \tag{1.1}$$

and the initial condition reads in this case as

$$x(0) = x_0 \in R^n. \tag{1.2}$$

Here $V$ is a $k$-linear Volterra operator (see below), which is defined in certain linear spaces of vector-valued stochastic processes.

By the $k$-linearity of the operator $V$ we mean the property

$$V(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 V x_1 + \alpha_2 V x_2,$$

which holds for all $\mathcal{F}_0$-measurable, bounded and scalar random values $\alpha_1$, $\alpha_2$ and all stochastic processes $x_1$, $x_2$ belonging to the domain of the operator $V$.

According to the paper [3] the following classes of linear stochastic equations can be rewritten in the form (1.2):

(a) *Systems of linear ordinary* (*i.e. non-delay*) *stochastic differential equations driven by an arbitrary semimartingale* (*in particular, systems of ordinary Itô equations*);

(b) *Systems of linear stochastic differential equations with discrete delays driven by a semimartingale* (*in particular, systems of Itô equations with discrete delays*);

(c) *Systems of linear stochastic differential equations with distributed delays driven by a semimartingale* (*in particular, systems of Itô equations with distributed delays*);

(d) *Systems of linear stochastic integro-differential equations driven by a semimartingale* (*in particular, systems of Itô integro-differential equations*);

(e) *Systems of linear stochastic functional difference equations driven by a semimartingale* (*in particular, systems of Itô functional difference equations*).

## 2 Lyapunov stability and $M$-stability

In this section we study different kinds of stochastic Lyapunov stability of the zero solution of the linear equation (1.1) with respect to the initial data (1.2). Let us start with the precise definitions.

**Definition 2.1.** The zero solution of the equation (1.1) is called

1. *weakly stable in probability* if for any $\varepsilon > 0$, $\delta > 0$ there is $\eta(\varepsilon,\delta) > 0$ such that $P\{\omega \in \Omega:\ |x(t,x_0)| > \varepsilon\} < \delta$ for all $|x_0| < \eta$ and $t \geq 0$;

2. *asymptotically weakly stable in probability* if it is weakly stable in probability and if, in addition, for any $\varepsilon > 0$ and all $x_0 \in R^n$ one has
$$P\big\{\omega \in \Omega:\ |x(t,x_0)| >\ \varepsilon\big\} \longrightarrow 0 \ \text{ as } \ t \to +\infty;$$

3. *stable in probability* if for any $\varepsilon, \delta > 0$ there is $\eta(\varepsilon,\delta) > 0$ such that
$$P\Big\{\omega \in \Omega:\ \sup_{t\geq 0} |x(t,x_0)| > \varepsilon\Big\} < \delta \ \text{ for all } \ |x_0| < \eta;$$

4. *asymptotically stable in probability* if it is stable in probability and if, in addition, for any $\varepsilon > 0$ and all $x_0 \in R^n$ one has $P\{\omega \in \Omega:\ |x(t,x_0)| > \varepsilon\} \to 0$ as $t \to +\infty$;

5. $p$-stable if for any $\varepsilon > 0$ there is $\eta(\varepsilon) > 0$ such that $|x_0| < \eta$ implies $E|x(t,x_0)|^p \leq \varepsilon$ for all $t \geq 0$;

6. asymptotically $p$-stable if it is $p$-stable and, in addition, $\lim\limits_{t\to+\infty} E|x(t,x_0)|^p = 0$ for all $x_0 \in R^n$;

7. exponentially $p$-stable if there exist positive constants $K$, $\beta$ such that the inequality
$$E|x(t,x_0)|^p \leq K|x_0|^p \exp\{-\beta t\}$$
holds true for all $t \geq 0$ and all $x_0 \in R^n$;

8. *stable with probability* 1 if $\sup\limits_{t\geq 0} |x(t,x_\nu)| \to 0$ with probability 1 whenever $|x_\nu| \to 0$ as $\nu \to +\infty$;

9. *asymptotically stable with probability* 1 if it is stable with probability 1 and if, in addition, $|x(t,x_0)| \to 0$ as $t \to +\infty$ for all $x_0 \in R^n$;

10. *strongly stable with probability* 1 if for any $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that
$$P\Big\{\omega \in \Omega : \sup_{t\ge 0} |x(t,x_0)| \le \varepsilon\Big\} = 1$$
whenever $|x_0| < \eta$;

11. *strongly asymptotically stable with probability* 1 if it is strongly stable with probability 1 and if, in addition, for any $\varepsilon > 0$, $x(t,x_0)$ tends to 0 with probability 1 as $t \to +\infty$ for all $x_0 \in R^n$.

**Remark 2.2.** The initial condition $x_0$ can also be random. In this case the norm of $x_0$ should be adjusted accordingly.

For brevity, we will also write "the equation (1.1) is stable" in a certain sense instead of "the zero solution of the equation (1.1) is stable" in this sense.

In the sequel the following linear spaces of stochastic processes will be used:

- $L^n(Z)$ consists of all predictable $n \times m$-matrix stochastic processes on $[0,+\infty)$, the rows of which are locally integrable w.r.t. the semimartingale $Z$ (see, e.g. [5]);
- $D^n$ consists of all $n$-dimensional stochastic processes on $[0,+\infty)$, which can be represented as
$$x(t) = x(0) + \int_0^t H(s)\,dZ(s),$$
where $x(0) \in R^n$, $H \in L^n(Z)$.

In addition to Lyapunov stability, one can consider the so-called "$M$-stability".

**Definition 2.3.** Let $x(\,\cdot\,,x_0)$ be the solution of the initial value problem (1.1)–(1.2) defined on $[0,\infty)$ and $M$ be a certain subspace of the space $D^n$. We say that the equation (1.1) is $M$-stable if $x(\,\cdot\,,x_0) \in M$ for any $x_0 \in R^n$.

The spaces below ("$M$-spaces") are crucial for studying the stochastic Lyapunov stabilities listed above.

- $M_0^\gamma = \Big\{x :\ x \in D^n$ such that for any $\delta > 0$ there is $K > 0$, for which $\sup\limits_{t\ge 0} P\big\{\omega :\ \omega \in \Omega,\ |\gamma(t)x(t)| > K\big\} < \delta\Big\}$;
- $\widehat{M}_0^\gamma = \Big\{x :\ x \in D^n$ such that for any $\delta > 0$ there is $K > 0$, for which $P\big\{\omega :\ \omega \in \Omega,\ \sup\limits_{t\ge 0}|\gamma(t)x(t)| > K\big\} < \delta\Big\}$;
- $M_p^\gamma = \big\{x :\ x \in D^n,\ \sup\limits_{t\ge 0} E|\gamma(t)x(t)|^p < \infty\big\}$ $(0 < p < \infty)$;
- $\widehat{M}_p^\gamma = \big\{x :\ x \in D^n,\ E\sup\limits_{t\ge 0} |\gamma(t)x(t)|^p < \infty\big\}$ $(0 < p < \infty)$;
- $M_\infty^\gamma = \widehat{M}_\infty^\gamma = \big\{x :\ x \in D^n,\ \operatorname*{ess\,sup}\limits_{(t,\omega)\in[0,+\infty[\,\times\Omega} |\gamma(t)x(t)| < \infty\big\}$;

For $\gamma(t) = 1$ $(t \ge 0)$ we also put

- $M_p^1 = M_p$ and $\widehat{M}_p^1 = \widehat{M}_p$ $(0 \le p \le \infty)$.

Let $B$ be a linear subspace of the space $L^n(Z)$ equipped with some norm $\|\cdot\|_B$. For a given positive and continuous function $\gamma(t)$ $(t \in [0,\infty))$ we define $B^\gamma = \{f : \ f \in B, \ \gamma f \in B\}$. The latter space becomes a linear normed space if we put $\|f\|_{B^\gamma} := \|\gamma f\|_B$. By this, the linear spaces $M_p^\gamma$, $\widehat{M}_p^\gamma$ become normed spaces if $1 \le p \le \infty$.

**Remark 2.4.** The above spaces can also be described as follows. Let $L_\infty(X)$ be the space consisting of all essentially bounded functions $g : [0,\infty) \to X$, while $\mathcal{L}_p(Y)$ be the space of measurable $(p = 0)$, $p$-integrable $(0 < p < \infty)$, essentially bounded $(p = \infty)$ functions $h : \Omega \to Y$, where $X$ and $Y$ are arbitrary separable Banach spaces. Then it is easy to see that $M_p^\gamma = L_\infty(\mathcal{L}_p(R^n))$ and $\widehat{M}_p^\gamma = \mathcal{L}_p(L_\infty(R^n))$ for all $0 \le p \le \infty$ and an arbitrary positive and continuous function $\gamma : [0,\infty) \to R$. This means that the above list of the $M$-spaces covers all possible combinations of Lebesgue spaces with respect to the variable $\omega \in \Omega$ and spaces of essentially bounded functions with respect to the variable $t \in [0,\infty)$. As we will see, this list covers also all types of stochastic Lyapunov stability described in Definition 2.1.

Below we use the following assumptions on a continuous positive function $\gamma(t)$, $t \in [0,\infty)$:

Property $\gamma 1$: the function $\gamma$ satisfies the conditions $\gamma(t) \ge \sigma$ $(t \in [0,+\infty))$, $\sigma > 0$ and $\lim\limits_{t\to+\infty} \gamma(t) = +\infty$.

Property $\gamma 2$: $\gamma(t) = \exp\{\beta t\}$ for some $\beta > 0$.

The theorem below describes relationships between the different kinds of the stochastic Lyapunov stability and the associated $M$-stabilities.

**Theorem 2.5.** *The following statements are valid for the equation* (1.1)*:*

1. *weak stability in probability is equivalent to the $M_0$-stability;*
2. *weak asymptotic stability in probability is equivalent to the $M_0^\gamma$-stability for some $\gamma$ satisfying Property $\gamma 1$;*
3. *stability in probability is equivalent to the $\widehat{M}_0$-stability;*
4. *if $0 < p < \infty$, then $p$-stability is equivalent to the $M_p$-stability;*
5. *if $0 < p < \infty$, then asymptotic $p$-stability is equivalent to the $M_p^\gamma$-stability for some $\gamma$ satisfying Property $\gamma 1$;*
6. *if $0 < p < \infty$, then exponential $p$-stability is equivalent to the $M_p^\gamma$-stability for some $\gamma$ satisfying Property $\gamma 2$;*
7. *stability with probability* 1 *is equivalent to the $\widehat{M}_0$-stability;*
8. *strong stability with probability* 1 *is equivalent to the $M_\infty$-stability;*
9. *strong asymptotic stability with probability* 1 *is equivalent to the $M_\infty^\gamma$-stability for some $\gamma$ satisfying Property $\gamma 1$.*

Using these results we can study relationships between different kinds of stochastic Lyapunov stability and $M$-stability.

**Corollary 2.6.** *Let $p \in [0,\infty]$. Then the following are valid for the stochastic functional differential equation* (1.1)*:*

1. $\widehat{M}_p$*-stability implies stability with probability* 1*;*
2. $\widehat{M}_p^{\gamma}$*-stability with* $\gamma$ *satisfying Property* $\gamma 1$ *implies asymptotic stability with probability* 1*.*
3. $\widehat{M}_\infty^{\gamma}$*-stability with* $\gamma$ *satisfying Property* $\gamma 1$ *implies strong asymptotic stability with probability* 1*.*

**Corollary 2.7.** *For the equation* (1.1) *we have:*

1. *if* $0 < q < p < \infty$*, then* $p$*-stability* (*resp. asymptotic, exponential* $p$*-stability*) *implies* $q$*-stability* (*resp. asymptotic, exponential* $q$*-stability*)*;*
2. *if* $0 < p < \infty$*, then* $p$*-stability* (*resp. asymptotic* $p$*-stability*) *implies weak stability in probability* (*resp. weak asymptotic stability in probability*)*;*
3. *stability in probability* (*resp. asymptotic stability in probability*) *implies weak stability with probability* 1 (*resp. weak asymptotic stability with probability* 1)*.*
4. *stability in probability is equivalent to stability with probability* 1*.*

The proof of the theorem and the corollaries as well as some applications can be found in [4].

## References

[1] N. V. Azbelev and P. M. Simonov, *Stability of Differential Equations with Aftereffect.* Stability and Control: Theory, Methods and Applications, 20. Taylor & Francis, London, 2003.

[2] R. I. Kadiev, Stability of solutions of stochastic functional differential equations. (Russian) *Habilitation thesis, Jekaterinburg*, 2000.

[3] R. I. Kadiev and A. V. Ponosov, Stability of linear stochastic functional-differential equations with constantly acting perturbations. (Russian) *Differentsial nye Uravneniya* **28** (1992), no. 2, 198–207; translation in *Differential Equations* **28** (1992), no. 2, 173–179.

[4] R. Kadiev and A. Ponosov, Lyapunov stability of the generalized stochastic pantograph equation. *J. Math.* **2018**, Art. ID 7490936, 9 pp.

[5] R. Sh. Liptser and A. N. Shiryayev, *Theory of Martingales.* Translated from the Russian by K. Dzjaparidze [Kacha Dzhaparidze]. Mathematics and its Applications (Soviet Series), 49. Kluwer Academic Publishers Group, Dordrecht, 1989.