On One Mixed Problem for One Class of Second Order Nonlinear Hyperbolic Systems with the Dirichlet and Poincare Boundary Conditions

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In the domain $D_T:\ 0<x<l,\ 0<t<T$ consider the following mixed problem

$$u_{tt}-u_{xx}+Au_x+Bu_t+Cu+f(x,t,u)=F(x,t),\ \ (x,t)\in D_T, \tag{1}$$

$$u(x,0)=\varphi(x),\quad u_t(x,0)=\psi(x),\ \ 0\le x\le l, \tag{2}$$

$$(Mu_x+Nu_t+Su)(0,t)=0,\quad u(l,t)=0,\ \ 0\le t\le T, \tag{3}$$

where A, B, C, M, N, S are given n-th order quadratic real matrix-functions; $f=(f_1,\dots,f_n)$, $F=(F_1,\dots,F_n)$, $\varphi=(\varphi_1,\dots,\varphi_n)$ and $\psi=(\psi_1,\dots,\psi_n)$ are given and $u=(u_1,\dots,u_n)$ is an unknown real vector-functions, $n\ge 2$.

Below we consider the problem (1)–(3) in a classical statement, when its regular solution is searched in the class $C^2(\overline{D}_T)$ and it is supposed that the problem data have corresponding smoothness and in the points $(0,0)$ and $(l,0)$ satisfy second order agreement conditions.

Divide the domain D_l, being a quadrat with the center in $O_1(\frac{l}{2},\frac{l}{2})$, into four triangles:

$$D_l^1:=OO_1O_2,\ \ D_l^2:=OO_1O_3,\ \ D_l^3:=O_2O_1O_4,\ \ D_l^4:=O_3O_1O_4,$$

where

$$O=(0,0),\ \ O_2=(l,0),\ \ O_3=(0,l),\ \ O_4=(l,l).$$

Assuming that

$$\det(M-N)(0,t)\neq 0,\ \ 0\le t\le l,$$

the problem (1)–(3) can be equivalently reduced to the Volterra type nonlinear integro-differential equation with respect to variable t by using the methods of Riemann matrices-functions and Laplacian invariants

$$u(x,t)=(Tu)(x,t),\ \ (x,t)\in D_l,$$

where

$$\begin{aligned}(Tu)(x,t)=\chi_1^1(x,t)\varphi(x-t)+\chi_2^1(x,t)\varphi(x+t)\\ +\int\limits_{P_1^1P_2^1}\big[\Lambda_1^1(x,t;\xi)\varphi(\xi)+\Lambda_2^1(x,t;\xi)\psi(\xi)\big]\,d\xi\\ +\int\limits_{D_{x,t}^1}K_1(x,t;\xi,\eta)\big[F(\xi,\eta)-f(\xi,\eta,u)\big]\,d\xi\,d\eta,\ \ P^1(x,t)\in D_l^1,\end{aligned} \tag{4}$$

where $P_1^1=(x-t,0)$, $P_2^1=(x+t,0)$, $D_{x,t}^1$ is a triangle $P_1^1P^1P_2^1$;

$$\begin{aligned}(Tu)(x,t)&=\chi_1^2(x,t)\varphi(0)+\chi_2^2(x,t)\varphi(t-x)+\chi_3^2(x,t)\varphi(t+x)\\&\quad+\int\limits_{OP_3^2}\big[\Lambda_1^2(x,t;\xi)\varphi(\xi)+\Lambda_2^2(x,t;\xi)\psi(\xi)\big]\,d\xi\\&\quad+\int\limits_{D_{x,t}^2}K_2(x,t;\xi,\eta)\big[F(\xi,\eta)-f(\xi,\eta,u)\big]\,d\xi\,d\eta,\ \ P^2(x,t)\in D_l^2,\end{aligned}\tag{5}$$

where $P_1^2=(0,t-x)$, $P_3^2=(t+x,0)$, $D_{x,t}^2$ is a quadrangle $OP_1^2P^2P_3^2$;

$$\begin{aligned}(Tu)(x,t)&=\chi_1^3(x,t)\varphi(x-t)+\chi_2^3(x,t)\varphi(2l-x-t)\\&\quad+\int\limits_{P_1^3O_1}\big[\Lambda_1^3(x,t;\xi)\varphi(\xi)+\Lambda_2^3(x,t;\xi)\psi(\xi)\big]\,d\xi\\&\quad+\int\limits_{D_{x,t}^3}K_3(x,t;\xi,\eta)\big[F(\xi,\eta)-f(\xi,\eta,u)\big]\,d\xi\,d\eta,\ \ P^3(x,t)\in D_l^3,\end{aligned}\tag{6}$$

where $P_1^3=(x-t,0)$, $P_3^3=(l,x+t-l)$, $D_{x,t}^3$ is a quadrangle $P^3P_1^3O_1P_3^3$;

$$\begin{aligned}(Tu)(x,t)&=\chi_1^4(x,t)\varphi(0)+\chi_2^4(x,t)\varphi(t-x)+\chi_3^4(x,t)\varphi(2l-x-l)\\&\quad+\int\limits_{OO_1}\big[\Lambda_1^4(x,t;\xi)\varphi(\xi)+\Lambda_2^4(x,t;\xi)\psi(\xi)\big]\,d\xi\\&\quad+\int\limits_{D_{x,t}^4}K_4(x,t;\xi,\eta)\big[F(\xi,\eta)-f(\xi,\eta,u)\big]\,d\xi\,d\eta,\ \ P^4(x,t)\in D_l^4,\end{aligned}\tag{7}$$

where $P_1^4=(0,t-x)$, $P_4^4=(l,x+t-l)$, $D_{x,t}^4$ is a quadrangle $P^4P_1^4OO_1P_4^4$; everywhere here χ_i^j, Λ_k^j and K_j, $i=1,2,3$, $k=1,2$, $j=1,2,3,4$ are well-known defined matrices.

For $f=0$ the formulas (4)–(7) give the solution of the posed linear problem in quadratures.

Notice, on supposition that $f\in C^1(D_\infty\times\mathbb{R})$ the problem (1)–(3) is locally always solvable, i.e. there exists a number $T_0=T_0(F,\varphi,\psi)>0$ such that for $T<T_0$ the problem is solvable in domain D_T. Besides, without additional requirements on the increment of nonlinearity of vector-function f and its structure, the problem (1)–(3) may not have a solution.