Productivity of Riccati Differential Equations

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1 Introduction

Consider the second order half-linear differential equation

$$(p(t)\varphi_\alpha(x'))' + q(t)\varphi_\alpha(x) = 0, \tag{E}$$

where α is a positive constant, $p(t)$ and $q(t)$ are positive continuous functions on $[a,\infty)$, $a \geq 0$, and $\varphi_\alpha(u) = |u|^\alpha \operatorname{sgn} u$, $u \in \mathbf{R}$.

We assume that equation (E) is nonoscillatory. Given a solution $x(t)$ of (E) we call the function $p(t)\varphi_\alpha(x'(t))$ the *quasi-derivative* of $x(t)$ and denote it by $Dx(t)$. If $u(t)$ and $v(t)$ are defined by

$$u(t) = \frac{Dx(t)}{\varphi_\alpha(x(t))} \quad \text{and} \quad v(t) = \frac{x(t)}{\varphi_{1/\alpha}(Dx(t))}\,,$$

then they satisfy the first order nonlinear differential equations

$$u' = -q(t) - \alpha p(t)^{-\frac{1}{\alpha}} |u|^{1+\frac{1}{\alpha}}, \tag{R1}$$

$$v' = p(t)^{-\frac{1}{\alpha}} + \frac{1}{\alpha}\, q(t)|v|^{1+\alpha}, \tag{R2}$$

for all large t. Equations (R1) and (R2) are referred to as the first and the second Riccati equations associated with (E). Note that (R2) has recently been discovered by Mirzov [3]. Conversely, suppose that (R1) and (R2) have solutions $u(t)$ and $v(t)$ defined for all large t, say on $[T,\infty)$. Such solutions $u(t)$ and $v(t)$ are termed *global solutions* of (R1) and (R2), respectively. Form the function $x(t)$ on $[T,\infty)$ by one of the following formulas which are collectively called the *reproducing formulas*

$$x(t) = \exp\bigg(\int\limits_T^t p(s)^{-\frac{1}{\alpha}} \varphi_{\frac{1}{\alpha}}(u(s))\,ds\bigg) \ \text{ or } \ x(t) = \exp\bigg(-\int\limits_t^\infty p(s)^{-\frac{1}{\alpha}} \varphi_{\frac{1}{\alpha}}(u(s))\,ds\bigg), \tag{1.1}$$

$$x(t)=\frac{1}{\varphi_{\frac{1}{\alpha}}(u(t))}\,\exp\bigg(-\frac{1}{\alpha}\int\limits_T^t\frac{q(s)}{u(s)}\,ds\bigg)\ \text{ or }\ x(t)=\frac{1}{\varphi_{\frac{1}{\alpha}}(u(t))}\,\exp\bigg(\frac{1}{\alpha}\int\limits_t^\infty\frac{q(s)}{u(s)}\,ds\bigg),$$

$$x(t)=\exp\bigg(\int\limits_T^t\frac{ds}{p(s)^{\frac{1}{\alpha}}v(s)}\bigg)\ \text{ or }\ x(t)=\exp\bigg(-\int\limits_t^\infty\frac{ds}{p(s)^{\frac{1}{\alpha}}v(s)}\bigg),$$

$$x(t)=v(t)\exp\bigg(-\frac{1}{\alpha}\int\limits_T^t q(s)\varphi_\alpha(v(s))\,ds\bigg)\ \text{ or }\ x(t)=v(t)\exp\bigg(\frac{1}{\alpha}\int\limits_t^\infty q(s)\varphi_\alpha(v(s))\,ds\bigg). \tag{1.2}$$

Then, $x(t)$ gives a nonoscillatory solution of equation (E) on $[T,\infty)$. This shows that equation (E) is nonoscillatory if and only if the Riccati equation (R1) (or (R2)) has a global solution.

We expect that the Riccati equations will be more productive in the sense that all nonoscillatory solutions of equation (E) can be reproduced from the global solutions of (R1) and/or (R2). As a result of our efforts made in [2] it has turned out that a majority of solutions of (E) can really be reproduced by solving (R1) and (R2) by means of fixed point techniques. Worthy of note is that both (R1) and (R2) are indispensable in the reproduction processes.

2 Main results

We need the following notations:

$$I_p=\int\limits_a^\infty p(t)^{-\frac{1}{\alpha}}\,dt,\quad I_q=\int\limits_a^\infty q(t)\,dt,$$

$$P_\alpha(t)=\int\limits_a^t p(s)^{-\frac{1}{\alpha}}\,ds\ \text{ if }\ I_p=\infty,\quad \pi_\alpha(t)=\int\limits_t^\infty p(s)^{-\frac{1}{\alpha}}\,ds\ \text{ if }\ I_p<\infty,$$

$$Q(t)=\int\limits_a^t q(s)\,ds\ \text{ if }\ I_q=\infty,\quad \rho(t)=\int\limits_t^\infty q(s)\,ds\ \text{ if }\ I_q<\infty.$$

Of crucial importance is the following classification of nonoscillatory solutions of (E). Let $x(t)$ be a solution of (E) such that $x(t)Dx(t)\neq 0$ on $[T,\infty)$. Both $x(t)$ and $Dx(t)$ are monotone and have the limits $x(\infty)=\lim\limits_{t\to\infty}x(t)$ and $Dx(\infty)=\lim\limits_{t\to\infty}Dx(t)$ in the extended real number system. The pair $(x(\infty),Dx(\infty))$, referred to as the terminal state of $x(t)$, is a decisive indicator of the asymptotic behavior at infinity of a solution $x(t)$ of (E). All possible types of terminal states of solutions $x(t)$ of (E) can be enumerated as follows.

(I) (The case where $I_p=\infty\wedge I_q<\infty$) (All solutions satisfy $x(t)Dx(t)>0$)

- I(i): $|x(\infty)|=\infty$, $0<|Dx(\infty)|<\infty$,
- I(ii): $|x(\infty)|=\infty$, $Dx(\infty)=0$,
- I(iii): $0<|x(\infty)|<\infty$, $Dx(\infty)=0$.

(II) (The case where $I_p<\infty\wedge I_q=\infty$) (All solutions satisfy $x(t)Dx(t)<0$)

- II(i): $0<|x(\infty)|<\infty$, $|Dx(\infty)|=\infty$,
- II(ii): $x(\infty)=0$, $|Dx(\infty)|=\infty$,

II(iii): $x(\infty) = 0$, $0 < |Dx(\infty)| < \infty$.

(III) (The case where $I_p < \infty \wedge I_q < \infty$)

$$\begin{aligned} \text{III(i)} &= \text{I(iii)}\ (x(t)Dx(t) > 0),\\ \text{III(ii)} &= \text{II(iii)}\ (x(t)Dx(t) < 0), \end{aligned}$$

III(iii): $0 < |x(\infty)| < \infty$, $0 < |Dx(\infty)| < \infty$) ($x(t)Dx(t) > 0$ or $x(t)Dx(t) < 0$).

The existence of solutions of the types I(i), I(iii), II(i) and II(iii) can be completely characterized.

Theorem 2.1. *Assume that $I_p = \infty \wedge I_q < \infty$.*

(i) (E) *has a solution of type* I(i) *if and only if* $\int\limits_a^\infty q(t)P_\alpha(t)^\alpha\,dt < \infty$.

(ii) (E) *has a solution of type* I(iii) *if and only if* $\int\limits_a^\infty p(t)^{-\frac{1}{\alpha}}\rho(t)^{\frac{1}{\alpha}}\,dt < \infty$.

Theorem 2.2. *Assume that $I_p < \infty \wedge I_q = \infty$.*

(i) (E) *has a solution of type* II(i) *if and only if* $\int\limits_a^\infty p(t)^{-\frac{1}{\alpha}}Q(t)^{\frac{1}{\alpha}}\,dt < \infty$.

(ii) (E) *has a solution of type* II(iii) *if and only if* $\int\limits_a^\infty q(t)\pi_\alpha(t)^\alpha\,dt < \infty$.

Only the proofs of the "if" parts of Theorem 2.1 are outlined.

Proof of the "if" part of Theorem 2.1-(i). Choose $T > a$ so that $\int\limits_T^\infty q(s)P_\alpha(s)^\alpha\,ds \le \alpha/(\alpha+1)2^{\alpha+1}$, define the set

$$\mathcal{V} = \big\{v \in C_{P_\alpha}[T,\infty):\ P_\alpha(t) \le v(t) \le 2P_\alpha(t),\ t \ge T\big\},$$

where $C_{P_\alpha}[T,\infty)$ denotes the Banach space of all continuous functions $w(t)$ on $[T,\infty)$ such that $|w(t)|/P_\alpha(t)$ is bounded with the norm $\|w\|_{P_\alpha} = \sup\{|w(t)|/P_\alpha(t):\ t \ge T\}$, and show that the integral operator given by

$$Gv(t) = P_\alpha(t) + \frac{1}{\alpha}\int\limits_T^t q(s)|v(s)|^{\alpha+1}\,ds,\ \ t \ge T,$$

is a contraction such that $\|Gv_1 - Gv_2\|_{P_\alpha} \le \frac{1}{2}\|v_1 - v_2\|_{P_\alpha}$ for any $v_1, v_2 \in \mathcal{V}$. Therefore, G has a unique fixed point $v \in \mathcal{V}$ which gives a solution $v(t)$ of (R2) on $[T,\infty)$ such that $v(t) \sim P_\alpha(t)$ as $t \to \infty$. With this $v(t)$ define $x(t)$ by the second formula in (1.2). Then, it is a solution of (E) satisfying $x(t) \sim P_\alpha(t)$ and $Dx(t) \sim 1$ as $t \to \infty$.

Proof of the "if" part of Theorem 2.1-(ii). Choose $T > a$ so that $\int\limits_T^\infty p(s)^{-\frac{1}{\alpha}}\rho(s)^{\frac{1}{\alpha}}\,ds \le 1/(\alpha+1)2^{1+\frac{1}{\alpha}}$ and consider the set

$$\mathcal{U} = \big\{v \in C_0[T,\infty):\ \rho(t) \le u(t) \le 2\rho(t),\ t \ge T\big\},$$

where $C_0[T,\infty)$ denotes the set of all continuous functions $w(t)$ on $[T,\infty)$ tending to zero as $t\to\infty$. It is a Banach space with the sup-norm $\|w\|_0=\sup\{|w(t)|:\ t\ge T\}$. Show that the integral operator given by

$$Fu(t)=\rho(t)+\alpha\int\limits_t^\infty p(s)^{-\frac{1}{\alpha}}|u(s)|^{1+\frac{1}{\alpha}}\,ds,\ \ t\ge T,$$

is a contraction such that $\|Fu_1-Fu_2\|_0\le\frac{1}{2}\|u_1-u_2\|_0$ for any $u_1,u_2\in\mathcal{U}$. Let $u\in\mathcal{U}$ be a unique fixed point of F. Then, it is a solution $u(t)$ of (R1) on $[T,\infty)$ such that $u(t)\sim\rho(t)$ as $t\to\infty$. Using this $u(t)$ define $x(t)$ according to the second reproducing formula of (1.1). Then, it is a positive solution of (E) satisfying $x(t)\sim 1$ and $Dx(t)\sim\rho(t)$ as $t\to\infty$.

Note that any solution of the type III(iii) of (E) in the case $I_p<\infty\wedge I_q<\infty$ can also be reproduced from a suitable solution of (R1) or (R2).

As for solutions of the types I(ii) and II(ii) of (E), often referred to as *intermediate solutions*, very little is known about their existence and asymptotic behavior at infinity. In [2] we have indicated several nontrivial cases of (E) whose intermediate solutions can actually be reproduced with the aid of (R1) and (R2).

Theorem 2.3.

(i) *Assume that $I_p=\infty\wedge I_q<\infty$. Equation* (E) *has an intermediate solution of the type* I(ii) *if*

$$\int\limits_a^\infty p(t)^{-\frac{1}{\alpha}}\rho(t)^{\frac{1}{\alpha}}\,dt=\infty,\quad \int\limits_a^\infty q(t)P_\alpha(t)^\alpha\,dt<\infty.$$

(ii) *Assume that $I_p<\infty\wedge I_q=\infty$. Equation* (E) *has an intermediate solution of the type* II(ii) *if*

$$\int\limits_a^\infty q(t)\pi_\alpha(t)^\alpha\,dt=\infty,\quad \int\limits_a^\infty p(t)^{-\frac{1}{\alpha}}Q(t)^{\frac{1}{\alpha}}\,dt<\infty.$$

Outline of proof of (i). Let any constant $A>1$ be given. Put $r(t)=\int\limits_t^\infty q(s)P_\alpha(s)^\alpha\,ds$ and choose $T>a$ so that $r(T)\le(A-1)^\alpha A^{-\alpha-1}$. Define the integral operator F by

$$Fu(t)=\rho(t)+\alpha\int\limits_t^\infty p(s)^{-\frac{1}{\alpha}}|u(s)|^{1+\frac{1}{\alpha}}\,ds,\ \ t\ge T,$$

and let it act on the set $\mathcal{U}$ defined by

$$\mathcal{U}=\big\{u\in C[T,\infty):\ \rho(t)\le u(t)\le Ar(t)P(t)^{-\alpha},\ t\ge T\big\},$$

which is a closed convex subset of the locally convex space $C[T,\infty)$.

Then, it can be shown that F is a continuous self-map of $\mathcal{U}$ sending $\mathcal{U}$ into a relatively compact subset of $C[T,\infty)$. Therefore, by the Schauder–Tychonoff fixed point theorem there exists a u in $\mathcal{U}$ such that $u=Fu$, which means that $u(t)$ is a global solution of (R1). With this $u(t)$ apply the first reproducing formula of (1.1) to construct a positive solution $x(t)$ of (E) on $[T,\infty)$. This is an intermediate solution of the type I(ii) since it is easily verified that $x(\infty)=\infty$ and $Dx(\infty)=0$.

Remark. Some of our results are already known; see e.g., [1]. However, our approach based on the Riccati equations makes the asymptotic analysis of equation (E) much easier and clearer.

References

[1] Z. Došlá and I. Vrkoč, On an extension of the Fubini theorem and its applications in ODEs. *Nonlinear Anal.* **57** (2004), no. 4, 531–548.

[2] J. Jaroš, T. Kusano and T. Tanigawa, Nonoscillatory solutions of planar half-linear differential systems: a Riccati equation approach. *Electron. J. Qual. Theory Differ. Equ.* (to appear).

[3] J. D. Mirzov, *Asymptotic Properties of Solutions of Systems of Nonlinear Nonautonomous Ordinary Differential Equations*. Folia Facultatis Scientiarium Naturalium Universitatis Masarykianae Brunensis. Mathematica, 14. Masaryk University, Brno, 2004.