On Additive Averaged Semi-Discrete Scheme for One Nonlinear Multi-Dimensional Integro-Differential Equation

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The present note is devoted to the nonlinear multi-dimensional integro-differential equation of parabolic type. The well-posedness of the initial-boundary value problem with first kind boundary condition and convergence of additive averaged semi-discrete scheme with respect to time variable are studied. The investigated equation is kind of natural generalization, on the one hand, of equations describing applied problems of mathematical physics and, on the other hand, of nonlinear parabolic equations considered, for example, in [14] and [18]. The studied equation is based on well-known Maxwell's system arising in mathematical simulation of electromagnetic field penetration into a substance [11].

Maxwell's system is complex and its investigation and numerical resolution still yield for special cases (see, for example, [9] and the references therein). In [3] the mentioned system was proposed in the integro-differential form. The literature on the questions of existence, uniqueness, and regularity of solutions to Maxwell's system and models of such integro-differential types is very rich. In [1–8, 12, 13], as well as in a number of other works the solvability of the initial-boundary value problems for this type integro-differential models in scalar cases are studied. The well-posedness of those problems in [1–8] are proved using a modified version of Galerkin's method and compactness arguments that are used in [14, 18] for investigation nonlinear elliptic and parabolic equations.

Let us note that the unique solvability and large time behavior of initial-boundary value problems for investigated in this note multi-dimensional integro-differential type equations at first are given in [4].

These questions and numerical resolution of initial-boundary value problems are discussed in many works as well (see, for example, [1–9, 12, 13, 16, 17] and the references therein).

Many authors study Rothe's type semi-discrete scheme with respect to time variable, semi-discrete schemes with spatial variable, finite element and finite difference approximations for a integro-differential models (see, for example, [5–10, 14, 16, 17] and the references therein).

It is very important to study decomposition analogs for the above-mentioned multi-dimensional integro-differential equation and systems too. At present there are some effective economic algorithms for solving the multi-dimensional problems (see, for example, [14, 15] and the references therein).

In this paper the existence and uniqueness of solutions of initial-boundary value problems is given. Main attention is paid to investigation of Rothe's type semi-discrete additive averaged scheme.

Let us formulate the studied problem. Let Ω be bounded domain in the n-dimensional Euclidean space R^n with sufficiently smooth boundary $\partial\Omega$. In the domain $Q=\Omega\times(0,T)$ of the variables $(x,t)=(x_1,x_2,\dots,x_n,t)$, where T is a positive constant, let us consider the following equation:

$$\frac{\partial U}{\partial t}-\sum_{i=1}^{n}\left\{\frac{\partial}{\partial x_i}\left[1+\int_0^t\left|\frac{\partial U}{\partial x_i}\right|^q d\tau\right]^p\left|\frac{\partial U}{\partial x_i}\right|^{q-2}\frac{\partial U}{\partial x_i}\right\}=f(x,t),\quad (x,t)\in Q, \tag{1}$$

with the homogeneous boundary and initial conditions:

$$U(x,t)=0,\ \ (x,t)\in\partial\Omega\times[0,T], \tag{2}$$

$$U(x,0)=0,\ \ x\in\overline{\Omega}, \tag{3}$$

where p, q are constants and f is a given function.

Principal characteristic peculiarity of the equation (1) is connected with the appearance of the nonlinear terms depended on the time integral in the coefficients with high order derivatives. These circumstances requires different discussions than it is usually necessary for the solution of local differential problems.

The problem (1)–(3) is similar to problems considered in [2, 4, 7, 12]. Unique solvability and discrete analogs of initial-boundary value problem for one-dimensional case of equation (1) are studied in [5]. Using modified version of Galerkin's method and compactness arguments as in [14,18] the following statement is obtained.

Theorem 1. *If $0<p\le 1$, $q\ge 2$, $f\in W_2^1(Q)$, $f(x,0)=0$, then there exists the unique solution U of problem (1)–(3) satisfying the following properties:*

$$U\in L_{pq+q}(0,T;\overset{\circ}{W}{}^1_{pq+q}(\Omega)),\quad \frac{\partial U}{\partial t}\in L_2(Q),$$

$$\sqrt{\psi}\,\frac{\partial U}{\partial x_j}\bigg(\Big|\frac{\partial U}{\partial x_i}\Big|^{\frac{q-2}{2}}\frac{\partial U}{\partial x_i}\bigg)\in L_2(Q),\quad \sqrt{T-t}\,\frac{\partial U}{\partial t}\bigg(\Big|\frac{\partial U}{\partial x_i}\Big|^{\frac{q-2}{2}}\frac{\partial U}{\partial x_i}\bigg)\in L_2(Q),\ \ i,j=1,\dots,n,$$

where $\psi\in C^\infty(\overline{\Omega})$, $\psi(x)>0$ for $x\in\Omega$; $\frac{\partial\psi}{\partial\nu}=0$ for $x\in\partial\Omega$ and ν is the outer normal of $\partial\Omega$.

Here we used usual L_p and W_p^k, $\overset{\circ}{W}{}_p^k$ Sobolev spaces.

Using the scheme of investigation as in [4] it is not difficult to get the results of exponential asymptotic behavior of solution as $t\to\infty$ of the initial-boundary value problems for the equation (1) with nonhomogeneous initial condition.

On $[0,T]$, let us introduce a net with mesh points denoted by $t_j=j\tau$, $j=0,1,\dots,J$, with $\tau=T/J$.

Coming back to problem (1)–(3), let us construct the following additive averaged Rothe's type scheme:

$$\eta_i\,\frac{u_i^{j+1}-u^j}{\tau}=\frac{\partial}{\partial x_i}\bigg[\bigg(1+\tau\sum_{k=1}^{j+1}\Big|\frac{\partial u_i^k}{\partial x_j}\Big|^q\bigg)^p\Big|\frac{\partial u_i^{j+1}}{\partial x_i}\Big|^{q-2}\frac{\partial u_i^{j+1}}{\partial x_i}\bigg]+f_i^{j+1}, \tag{4}$$

with the homogeneous boundary and initial $u_i^0=u^0=0$ conditions, where $u_i^j(x)$, $i=1,\dots,n$, $j=0,1,\dots,J-1$ are solutions of the problems (4). The notations in (4) are as follows:

$$u^j(x)=\sum_{i=1}^n\eta_iu_i^j(x),\quad \sum_{i=1}^n\eta_i=1,\ \ \eta_i>0,\quad \sum_{i=1}^n f_i^{j+1}(x)=f^{j+1}(x)=f(x,t_{j+1}),$$

where u^j denotes approximation of an exact solution U of the problem (1)–(3) at t_j. We use usual norm $\|\cdot\|$ of the space $L_2(\Omega)$.

Theorem 2. *If problem (1)–(3) has sufficiently smooth solution, then the solution of the problem (4) with homogeneous initial and boundary conditions converges to the solution of the problem (1)–(3) and the following estimate is true*

$$\|U^j-u^j\|=O(\tau^{1/2}),\ \ j=1,\dots,J.$$

Let us note that the results analogous to Theorem 2 for the following integro-differential models are obtained in the works [6–8]:

$$\frac{\partial U}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[\left(1 + \int_0^t \left| \frac{\partial U}{\partial x_i} \right|^2 d\tau \right) \frac{\partial U}{\partial x_i} \right] = f(x,t),$$

and

$$\frac{\partial U}{\partial t} - \sum_{i=1}^{n} \left(1 + \int_\Omega \int_0^t \left| \frac{\partial U}{\partial x_i} \right|^2 dx\, d\tau \right) \frac{\partial^2 U}{\partial x_i^2} = f(x,t).$$

It was mentioned in [7] that it is very important to construct and investigate (4) type semi-discrete additive schemes for more general type nonlinearities. The purpose of this work was to expand the previously studied cases. Thus, in this note we studied more wide class of nonlinearity.

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