Baer's Classification of Characteristic Exponents in the Full Perron's Effect of Their Value Change

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We consider the linear differential system

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^2, \ \ t \geq 0, \tag{1}$$

with a bounded continuously differentiable matrix of coefficients $A(t)$ and with negative characteristic exponents $\lambda_1(A) \leq \lambda_2(A) < 0$. This system is a linear approximation for the nonlinear system

$$\dot{y} = A(t)y + f(t, y), \ \ y = (y_1, y_2) \in \mathbb{R}^2, \ \ t \geq 0. \tag{2}$$

In addition, the so-called m-perturbation of $f(t, y)$ is continuously differentiable in its arguments $t \geq 0$ and $y_1, y_2 \in \mathbb{R}$ and has the order $m > 1$ of smallness in some neighborhood of the origin and admissible growth outside of it:

$$\|f(t, y)\| \leq C_f \|y\|^m, \ \ m > 1, \ \ y \in \mathbb{R}^2, \ \ t \geq 0, \tag{3}$$

where C_f is a positive constant.

Perron's effect [28], [27, pp. 50, 51] of sign and value change in characteristic exponents claims the existence of such system (1) with the negative Lyapunov exponents and 2-perturbation (3) that all nontrivial solutions of the perturbed system (2) turn out to be infinitely extendable and have finite Lyapunov exponents equal to:

1) the negative higher exponent λ_2 of the initial system (1) for the solutions starting at the initial moment on the axis $y_1 = 0$ (that allows one to consider Perron's effect incomplete);

2) any one positive value for all the rest solutions (calculated in [10, pp. 13–15]).

In our works [3–8, 11–24], we obtained various versions of the full Perron's effect when all nontrivial solutions of the nonlinear system (2) with m-perturbation (3) are infinitely extendable (this is not so in a general case) and have finite positive Lyapunov exponents for negative exponents of the system of linear approximation (1). These versions correspond to: different types of the set $\lambda(A, f) \subset (0, +\infty)$ of characteristic Lyapunov exponents of all nontrivial solutions of the perturbed system (2), distribution of those solutions with respect to the exponents from the set $\lambda(A, f)$ and, finally, an arbitrary order of systems (1) and (2). In particular, in our last works [14, 15], we obtained a continual version of the full Perron's effect with an arbitrarily given segment, a set $\lambda(A, f) \subset (0, +\infty)$ of characteristic exponents of the perturbed system (2).

In the full Perron's effect, the question dealing, in particular, with a most general type of the set $\lambda(A,f)$ of characteristic exponents (of all nontrivial solutions) of the perturbed system (2), i.e., the question on a full description of that set, remains still open. The aim of the present work is to establish that in the full Perron's effect of value change in characteristic exponents their set $\lambda(A,f)$ is the Suslin's one [2, pp. 97, 98, 192], realizing thus the first stage of the above description. Towards this end, it will be proved that within the framework of the effect under consideration the characteristic exponent

$$\lambda[y(\cdot,y_0)] \equiv \varlimsup_{t\to+\infty} \frac{1}{t}\,\ln\|y(t,y_0)\|$$

of every nontrivial solution $y(t,y_0)$ of system (2), being the function of the initial vector $y_0 = y(0,y_0) \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, is the function of the second Bare's class [2, p. 248]. Thus its set of values

$$\Lambda(A,f) \equiv \big\{\lambda[y(\,\cdot\,,y_0)] : y_0 \in \mathbb{R}^2 \setminus \{\mathbf{0}\}\big\}$$

belongs to the class of Suslin's sets [2, pp. 97,98, 192].

The perturbed differential system (2) realizing the full Perron's effect of values change, whose all nontrivial solutions take their origin in some neighbourhood of its zero solution and have, by the definition, positive exponents, may be called exponentially nonstable. In an opposite case, in no way connected with the Perron's effect, when the exponentially stable system (1) is such that any system (2) with m-perturbation f is likewise exponentially stable, we studied the set [9] $\Lambda_0(A,f) = \bigcap\limits_{\rho>0} \Lambda_\rho(A,f)$, where $\Lambda_\rho(A,f)$ is a set of Lyapunov's exponents of nontrivial solutions of system (2), emanating for $t=0$ from the ρ-neighbourhood of zero. For the set $\Lambda_0(A,f) \subset (-\infty,0)$, we obtained the following results. In [9], for an arbitrary segment $[\alpha,\beta] \subset (-\infty,0)$, we constructed the system (2) for which $\Lambda_0(A,f) = [\alpha,\beta]$. In [29], these constructions were extended to the sets $\Lambda_0(A,f) \subset (-\infty,0)$ consisting of a countable number of connectedness components. Finally, in [1], the family of sets $\Lambda_0(A,f)$ is described completely; it consists of bounded Suslin's sets of the negative semi-axis whose exact upper bound is negative.

The essentials of the Baer's classification of Lyapunov exponents and other asymptotic characteristics of solutions of parametric differential systems, as the functions of a parameter, were laid by V. M. Millionshchikov. Its subsequent development is connected with the works of M. I. Rakhimberdiev, I. N. Sergeyev, E. A. Barabanov, A. N. Vetokhin, V. V. Bykov and their pupils.

We will consider a more general, as compared with (2), the n-dimwnsional differential system

$$\dot y = F(t,y), \;\; y \in \mathbb{R}^n, \;\; t \ge 0, \tag{4}$$

with a continuously differentiable in its arguments $t > 0$ and $y_1,\dots,y_n \in \mathbb{R}$ right-hand side $F(t,y)$ satisfying the condition $F(t,\mathbf{0}) \equiv \mathbf{0}$, $t \ge 0$.

The following theorem is valid.

Theorem. *Let all nontrivial solutions $y(t,y_0)$ of system (4) be infinitely extendable and have finite characteristic exponents. Then the characteristic exponent $\lambda[y(\,\cdot\,,y_0)]$ of those solutions is the function of the 2nd Baer's class of their initial vectors $y_0 \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.*

Getting back to the full Perron's effect of value change in negative characteristic exponents of the system of linear approximation (1), for the whole set $\Lambda(A,f)$ of positive Lyapunov exponents of all nontrivial solutions of the perturbed system (2), we obtain the following

Corollary. *Let all nontrivial solutions $y(t,y_0)$, $y_0 \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ of system (2) be infinitely extendable and have finite positive Lyapunov exponents. Then the characteristic exponent $\lambda[y(\,\cdot\,,y_0)]$ of those solutions is the function of the 2nd Baer' class of their initial values $y_0 \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, whereas the whole set $\Lambda(A,f)$ of exponents of nontrivial solutions is Suslin's one.*

Remark 1. The above corollary is likewise valid for the n-dimensional analogue of the full Perron's effect.

Remark 2. In addition to the monograph by G. A. Leonov [27] the works due to V. V. Kozlov [25, 26] had a stimulating influence on our investigations of Perron's effect of sign and value change in characteristic exponents.

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