Theorems on Functional Differential Inequalities

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Consider the system of functional differential inequalities

$$\mathcal{D}(\sigma(t))\big[u'(t)-\ell(u)(t)\big]\geq 0 \ \text{ for a.e. } \ t\in[a,b], \tag{1}$$

$$\varphi(u)\geq 0, \tag{2}$$

where $\ell : C([a,b];\mathbb{R}^n)\to L([a,b];\mathbb{R}^n)$ is a linear bounded operator, $\varphi : C([a,b];\mathbb{R}^n)\to\mathbb{R}^n$ is a linear bounded functional, $\sigma=(\sigma_i)_{i=1}^n$, $\sigma_i:[a,b]\to\{-1,1\}$ are functions of bounded variation, and $\mathcal{D}(\sigma(t))=\operatorname{diag}(\sigma_1(t),\dots,\sigma_n(t))$. In the present contribution, we establish conditions guaranteeing that every absloutely continuous vector-valued function u satisfying (1) and (2) admits also the inequality $u(t)\geq 0$ for $t\in[a,b]$. For this purpose we will need the following notation and definitions.

$\mathbb{R}$ is a set of all real numbers, $\mathbb{R}_+=[0,+\infty[$, $\mathbb{R}^n$ is a space of n-dimensional column vectors $x=(x_i)_{i=1}^n$ with elements $x_i\in\mathbb{R}$ $(i=1,\dots,n)$, $\mathbb{R}^{n\times n}$ is a space of $n\times n$-matrices $X=(x_{ij})_{i,j=1}^n$ with elements $x_{ij}\in\mathbb{R}$ $(i,j=1,\dots,n)$, $\mathbb{R}_+^n$ and $\mathbb{R}_+^{n\times n}$ are sets of non-negative column vectors and matrices, respectively. The inequalities between vectors and matrices are understood componentwise. If 0 and 1 are used as vectors, then 0 is a zero column vector and 1 is a column vector with all components equal to one; δ_{ik} is the Kronecker's symbol; X^{-1} is the inverse matrix to X; $r(X)$ is the spectral radius of the matrix X; Θ is a zero matrix.

$C([a,b];\mathbb{R}^n)$ is a Banach space of continuous vector-valued functions $x=(x_i)_{i=1}^n:[a,b]\to\mathbb{R}^n$ endowed with the norm

$$\|x\|_C=\max\bigg\{\sum_{i=1}^n|x_i(t)|:\ t\in[a,b]\bigg\}.$$

$AC([a,b];\mathbb{R}^n)$ is a set of absolutely continuous vector-valued functions $x:[a,b]\to\mathbb{R}^n$.

$L([a,b];\mathbb{R}^n)$ is a Banach space of Lebesgue integrable vector-valued functions $p=(p_i)_{i=1}^n$: $[a,b]\to\mathbb{R}^n$ endowed with the norm

$$\|p\|_L=\int_a^b\sum_{i=1}^n|p_i(s)|\,ds.$$

$\mathcal{L}_{ab}^n$ is a set of linear bounded operators $\ell:C([a,b];\mathbb{R}^n)\to L([a,b];\mathbb{R}^n)$.

$\mathcal{C}_{ab}^{n,*}$ is a set of linear bounded functionals $\varphi:C([a,b];\mathbb{R}^n)\to\mathbb{R}^n$.

For any $\ell\in\mathcal{L}_{ab}^n$, the operators $\ell_i:C([a,b];\mathbb{R}^n)\to L([a,b];\mathbb{R})$ and $\ell_{ik}:C([a,b];\mathbb{R})\to L([a,b];\mathbb{R})$ $(i,k=1,\dots,n)$ are defined as follows:

- for any $v\in C([a,b];\mathbb{R}^n)$, $\ell_i(v)$ is the i-th component of the vector-valued function $\ell(v)$;
- for any $z\in C([a,b];\mathbb{R})$ we put $\ell_{ik}(z)=\ell_i(\widehat{z})$, where $\widehat{z}=(\delta_{ik}z)_{i=1}^n$.

For any functional $\varphi\in\mathcal{C}_{ab}^{n,*}$ we define the functionals $\varphi_i : C([a,b];\mathbb{R}^n)\to\mathbb{R}$ and φ_{ik} : $C([a,b];\mathbb{R})\to\mathbb{R}$ in a similar way. Moreover, we put $\Phi=\big(\varphi_{ik}(1)\big)_{i,k=1}^n$.

Definition 1. An operator $\ell \in \mathcal{L}_{ab}^n$ is said to be *σ-positive* if the relation

$$\mathcal{D}(\sigma(t))\ell(u)(t) \geq 0 \ \text{ for a.e. } \ t \in [a,b] \tag{3}$$

is fulfilled whenever $u \in C([a,b];\mathbb{R}^n)$ is such that

$$u(t) \geq 0 \ \text{ for } \ t \in [a,b] \tag{4}$$

holds. A set of σ-positive operators is denoted by $\mathcal{P}_{ab}^n(\sigma)$.

Definition 2. We will say that an operator $\ell \in \mathcal{L}_{ab}^n$ belongs to the set $\mathcal{P}_{ab}^{n,+}(\sigma)$ if the relation (3) is fulfilled whenever $u \in AC([a,b];\mathbb{R}^n)$ is such that (4) and

$$\mathcal{D}(\sigma(t))u'(t) \geq 0 \ \text{ for a.e. } \ t \in [a,b] \tag{5}$$

hold.

Remark 1. Obviously, $\mathcal{P}_{ab}^n(\sigma) \subsetneq \mathcal{P}_{ab}^{n,+}(\sigma)$.

Definition 3. We will say that a pair of operators $(\ell,\varphi) \in \mathcal{L}_{ab}^n \times \mathcal{C}_{ab}^{n,*}$ belongs to the set $\mathcal{S}_{ab}^n(\sigma)$ if every function $u \in AC([a,b];\mathbb{R}^n)$ satisfying (1), (2) admits also (4).

Remark 2. Obviously, if $(\ell,\varphi) \in \mathcal{S}_{ab}^n(\sigma)$, then the problem

$$u'(t) = \ell(u)(t) + q(t) \ \text{ for a.e. } \ t \in [a,b], \quad \varphi(u) = c$$

has a unique solution $u \in AC([a,b];\mathbb{R}^n)$ for every $q \in L([a,b];\mathbb{R}^n)$ and $c \in \mathbb{R}^n$, and this solution is non-negative if $\mathcal{D}(\sigma(t))q(t) \geq 0$ for a. e. $t \in [a,b]$ and $c \geq 0$.

In the formulation of the main results, the inclusion $(0,\varphi) \in \mathcal{S}_{ab}^n(\sigma)$ is used. Therefore, we present here some basic implication of this inclusion.

Proposition 1. *Let* $(0,\varphi) \in \mathcal{S}_{ab}^n(\sigma)$. *Then*

(i) $\det \Phi \neq 0$,

(ii) $\Phi^{-1} \geq \Theta$.

Proposition 2. *Let* $(0,\varphi) \in \mathcal{S}_{ab}^n(\sigma)$ *and let* $u \in AC([a,b];\mathbb{R}^n)$ *satisfy* (5). *Then*

$$u(t) \geq \Phi^{-1}\varphi(u) \ \textit{ for } \ t \in [a,b].$$

Main results

Theorem 1. *Let* $\ell \in \mathcal{P}_{ab}^n(\sigma)$, $(0,\varphi) \in \mathcal{S}_{ab}^n(\sigma)$. *Then* $(\ell,\varphi) \in \mathcal{S}_{ab}^n(\sigma)$ *iff there exists* $\gamma \in AC([a,b];\mathbb{R}^n)$ *such that*

$$\mathcal{D}(\sigma(t))\big[\gamma'(t) - \ell(\gamma)(t)\big] \geq 0 \ \textit{ for a.e. } \ t \in [a,b],$$
$$\gamma(t) > 0 \ \textit{ for } \ t \in [a,b], \quad \Phi^{-1}\varphi(\gamma) > 0.$$

Proof. Necessity: If $(\ell,\varphi) \in \mathcal{S}_{ab}^n(\sigma)$, then according to Remark 2 the problem

$$u'(t) = \ell(u)(t) + \ell(1)(t) \ \text{ for a.e. } \ t \in [a,b], \quad \varphi(u) = 0$$

is uniquely solvable. Moreover, $u(t) \geq 0$ for $t \in [a,b]$. Put $\gamma(t) = u(t) + 1$ for $t \in [a,b]$. Then

$$\mathcal{D}(\sigma(t))\big[\gamma'(t) - \ell(\gamma)(t)\big] = 0 \text{ for a.e. } t \in [a,b],$$
$$\gamma(t) > 0 \text{ for } t \in [a,b], \quad \Phi^{-1}\varphi(\gamma) = \Phi^{-1}(\varphi(u) + \Phi \cdot 1) > 0.$$

Sufficiency: Let u satisfy (1), (2) with $u_j(t_j) < 0$ for some $j \in \{1,\dots,n\}$ and $t_j \in [a,b]$. Put

$$\lambda_i = \max\Big\{ -\frac{u_i(t)}{\gamma_i(t)} : t \in [a,b]\Big\} \ \ (i = 1,\dots,n)$$

and let

$$\lambda = \max\big\{\lambda_1,\dots,\lambda_n\big\} > 0.$$

Then $w(t) \overset{def}{=} \lambda\gamma(t) + u(t) \geq 0$ for $t \in [a,b]$, and there exist $i_0 \in \{1,\dots,n\}$ and $t_0 \in [a,b]$ such that $w_{i_0}(t_0) = \lambda\gamma_{i_0}(t_0) + u_{i_0}(t_0) = 0$. Consequently,

$$\mathcal{D}(\sigma(t))w'(t) \geq \mathcal{D}(\sigma(t))\ell(w)(t) \geq 0 \text{ for a.e. } t \in [a,b].$$

According to Proposition 2,

$$w(t) \geq \Phi^{-1}\varphi(w) = \Phi^{-1}(\lambda\varphi(\gamma) + \varphi(u)) > 0,$$

a contradiction. ◻

Theorem 2. *Let ℓ admit the representation $\ell = \ell^+ - \ell^-$ where $\ell^+, \ell^- \in \mathcal{P}^n_{ab}(\sigma)$. Let, moreover,*

$$\ell \in \mathcal{P}^{n,+}_{ab}(\sigma), \ \ (\ell^+, \varphi) \in \mathcal{S}^n_{ab}(\sigma), \ \ (0, \varphi) \in \mathcal{S}^n_{ab}(\sigma).$$

Then $\ell \in \mathcal{S}^n_{ab}(\sigma)$.

Proof. Let u satisfy (1), (2). According to Remark 2 there exists a unique solution x to the problem

$$x'(t) = \mathcal{D}(\sigma(t))\big[\mathcal{D}(\sigma(t))u'(t)\big]_- \text{ for a.e. } t \in [a,b], \quad \varphi(x) = 0.$$

Moreover, we have $x(t) \geq 0$ for $t \in [a,b]$. Put $w(t) = u(t) + x(t)$ for $t \in [a,b]$. Then $w(t) \geq u(t)$ for $t \in [a,b]$,

$$\mathcal{D}(\sigma(t))w'(t) = \big[\mathcal{D}(\sigma(t))u'(t)\big]_+ \geq 0 \text{ for a.e. } t \in [a,b], \quad \varphi(w) \geq 0.$$

Thus, $w(t) \geq 0$ for $t \in [a,b]$. Let $A_i = \{t \in [a,b] : w_i'(t) = u_i'(t)\}$ and put

$$q(t) \overset{def}{=} \mathcal{D}(\sigma(t))\big[u'(t) - \ell(u)(t)\big] \text{ for a.e. } t \in [a,b].$$

Then, for every $i \in \{1,\dots,n\}$, we have

$$\sigma_i(t)w_i'(t) = \begin{cases} \sigma_i(t)u_i'(t) = \sigma_i(t)\displaystyle\sum_{k=1}^n \big[\ell^+_{ik}(u_k)(t) - \ell^-_{ik}(u_k)(t)\big] + q_i(t) \\ \qquad\qquad \leq \sigma_i(t)\displaystyle\sum_{k=1}^n \big[\ell^+_{ik}(w_k)(t) - \ell^-_{ik}(u_k)(t)\big] + q_i(t) \text{ for } t \in A_i, \\ 0 \qquad\qquad \text{for a.e. } t \in [a,b] \setminus A_i. \end{cases}$$

On the other hand,

$$\mathcal{D}(\sigma(t))\big[\ell^+(w)(t) - \ell^-(u)(t)\big] + q(t) \geq \mathcal{D}(\sigma(t))\ell(w)(t) + q(t) \geq 0 \text{ for a.e. } t \in [a,b].$$

Consequently,

$$\mathcal{D}(\sigma(t))\big[w'(t) - \ell^+(w)(t)\big] \le -\mathcal{D}(\sigma(t))\ell^-(u)(t) + q(t) \ \text{ for a.e. } t \in [a,b].$$

Put $z(t) = u(t) - w(t)$ for $t \in [a,b]$. Then

$$\mathcal{D}(\sigma(t))\big[z'(t) - \ell^+(z)(t)\big] \ge 0 \ \text{ for a.e. } t \in [a,b], \quad \varphi(z) = 0,$$

and so $z(t) \ge 0$ for $t \in [a,b]$, i.e. $u(t) \ge w(t) \ge 0$ for $t \in [a,b]$. □

As a consequences of the main results we formulate corollaries in the case when σ is a constant function. Therefore, in what follows we assume that $\sigma(t) = (\sigma_i)_{i=1}^n$ for $t \in [a,b]$ with $\sigma_i \in \{-1, 1\}$. First consider the system with deviating arguments

$$\sigma_i\Big[u_i'(t) - \sum_{k=1}^n \big(p_{ik}(t)u_k(\tau_{ik}(t)) - g_{ik}(t)u_k(\mu_{ik}(t))\big)\Big] \ge 0 \ \text{ for a.e. } t \in [a,b], \tag{6}$$

$$u_i(a) \ge 0 \ \text{ if } \sigma_i = 1, \quad u_i(b) \ge 0 \ \text{ if } \sigma_i = -1, \tag{7}$$

where $\sigma_i p_{ik}, \sigma_i g_{ik} \in L([a,b];\mathbb{R}_+)$, $\tau_{ik}, \mu_{ik} : [a,b] \to [a,b]$ are measurable functions.

Corollary 1. *Let*

$$\sigma_i\big(p_{ik}(t) - g_{ik}(t)\big) \ge 0, \quad \sigma_i\sigma_k g_{ik}(t)\big(\tau_{ik}(t) - \mu_{ik}(t)\big) \ge 0 \ \textit{ for a.e. } t \in [a,b].$$

Let, moreover, there exist $A = \big(a_{ik}\big)_{i,k=1}^n \in \mathbb{R}_+^{n\times n}$ *such that* $r(A) < 1$ *and*

$$\int_a^b \bigg(\sigma_i\big(p_{ik}(t) - g_{ik}(t)\big) + \sigma_i g_{ik}(t) \int_{\mu_{ik}(t)}^{\tau_{ik}(t)} \sum_{j=1}^n p_{kj}(s)\,ds\bigg)\,dt \le a_{ik}.$$

Then every $u \in AC([a,b];\mathbb{R}^n)$ *that satisfies* (6), (7) *is non-negative.*

The next corollary deals with the second-order differential inequality with deviations together with mixed boundary value conditions

$$u''(t) \le -p(t)u(\tau(t)) + g(t)u(\mu(t)) \ \text{ for a.e. } t \in [a,b], \ \ u(a) \ge 0, \ \ u'(b) \ge 0. \tag{8}$$

Here $p, g \in L([a,b];\mathbb{R}_+)$ and $\tau, \mu : [a,b] \to [a,b]$ are measurable functions.

Corollary 2. *Let*

$$\tau(t) \le t, \ \ p(t) \ge g(t), \ \ g(t)(\tau(t) - \mu(t)) \ge 0 \ \textit{ for a.e. } t \in [a,b].$$

Let, moreover, there exists $\lambda_1, \lambda_2 \in \mathbb{R}_+$ *such that*

$$\int_0^{+\infty} \frac{ds}{\lambda_1 + \lambda_2 s + s^2} \ge b - a,$$

$$p(t) - g(t) + g(t)(\tau(t) - \mu(t)) \int_{\tau(t)}^t p(s)\,ds + g(t) \int_{\mu(t)}^{\tau(t)} (s - \mu(t))p(s)\,ds \le \lambda_1 \ \textit{ for a.e. } t \in [a,b],$$

$$g(t)(\tau(t) - \mu(t)) \le \lambda_2 \ \textit{ for a.e. } t \in [a,b],$$

and at least one of the last three inequalities is strict. Then every $u \in AC^1([a,b];\mathbb{R})$ *that satisfies* (8) *is non-negative and nondecreasing.*