Dulac–Cherkas Method for Detecting Exact Number of Limit Cycles for Planar Autonomous Systems

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We consider the autonomous system of differential equations on the real plane

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y), \ \ (x,y) \in \Omega \subset R^2, \ \ P(x,y), Q(x,y) \in C^1(\Omega). \tag{1}$$

The Dulac criterion [1, p. 226], [8,9] is one of the ways to obtain nonlocal solution of the problem of counting and localizing the limit cycles [7] of system (1). However, there is no regular methods for finding a connected domain Ω of localization of the limit cycles and for constructing the Dulac function in this domain. Therefore, this criterion was predominantly used for proving the absence of limit cycles in a simply-connected domain Ω or the existence of at most one limit cycle in a doubly connected domain Ω. L. A. Cherkas [2] suggested to develop the Dulac criterion and to construct a special Dulac function in a connected domain Ω where the number and localization of limit cycles can be determined by using transversal curves that correspond to such function. This criterion is referred to as the Dulac–Cherkas criterion and allows one to derive an upper bound for the number of limit cycles for many classes of systems (1) [3, 4]. Additional research is needed to produce an exact estimate for the number of limit cycles but it is possible only in separate cases. Thus, our aim here is to present approaches developed by us to obtaining an exact nonlocal estimate for the number of limit cycles that surround one equilibrium point of system (1) and localizing these cycles. The Dulac or Dulac–Cherkas methods are applied sequentially two times to find closed transversal curves that divide the domain Ω in doubly connected subdomains surrounding the equilibrium point such that the system (1) has exactly one limit cycle in each of them.

The Dulac–Cherkas method as a generalization of the Dulac criterion consists in finding the Dulac–Cherkas function $\Psi(x,y)$ [3, p. 199].

Definition 1. A function $\Psi \in C^1(\Omega, R)$ is called as the Dulac–Cherkas function of system (1) in a domain Ω if there exists such a real number $k \neq 0$ that the following condition holds

$$\Phi(x,y) = k\Psi \operatorname{div} X + \frac{\partial \Psi}{\partial x}\, P + \frac{\partial \Psi}{\partial y}\, Q \geqslant 0 \ \ (\leqslant 0), \ \ \forall\, (x,y) \in \Omega \subset R^2, \tag{2}$$

where X is a vector field defined by system (1).

Remark 1. In inequality (2), it is usually assumed [1, p. 226], [2,8,9], [3, p. 68] that the function Φ can be zero on a set of the zero measure in the domain Ω, with no closed curve in this set being a limit cycle of system (1). However, Cherkas et al. [3, p. 312] showed that this requirement can be relaxed and replaced with the condition that the curve defined by the equation $\Phi(x,y) = 0$ is transversal.

Remark 2. If Ψ is a Dulac–Cherkas function of system (1) in the domain Ω, then $B = |\Psi|^{\frac{1}{k}}$ is a Dulac function in each subdomain Ω_i, where $\Psi > 0 (< 0)$, while any limit cycle Γ of system (1) that exists in Ω is rough and stable (unstable) under the condition that $k\Phi\Psi < 0 (> 0)$ on Γ.

To localize the limit cycles in the domain Ω, we introduce a set $W = \{(x,y) \in \Omega : \ \Psi(x,y) = 0\}$, that is transversal for the vector field X under condition (2) and is not intersected by the limit cycles of system (1).

The following assertion was proved in the monograph [3, p. 205].

Theorem 1 (the Dulac–Cherkas criterion)**.** *Suppose that in a connected domain Ω system* (1) *has the unique anti-saddle point of rest O, while Ψ is the Dulac–Cherkas function of system* (1) *with $k < 0$ in the domain Ω, where the set W consists of s mutually embedded ovals ω_i surrounding the point O. Then, system* (1) *has exactly one limit cycle in each of the $s-1$ ring-shaped subdomains Ω_i that are bounded by neighboring ovals ω_i and ω_{i+1} and can have at most s limit cycles in the domain Ω in total.*

The monograph [3] contains different ways for constructing the Dulac–Cherkas function which allows to estimate the upper bound for the number of limit cycles by using Theorem 1.

In cases where this approach is difficult to be implemented, it was suggested in [3, p. 334] to construct the Dulac function in the form of the product

$$B = |\Psi(x,y)|^{\frac{1}{k}} |\widetilde{\Psi}(x,y)|^{\frac{1}{\widetilde{k}}}, \ \ k, \widetilde{k} \in R, \ \ k\widetilde{k} \neq 0, \ \ \Psi, \widetilde{\Psi} \in C^1(\Omega). \tag{3}$$

Theorem 2. *A function B of the form* (3) *is the Dulac function of system* (1) *in the domain Ω if the following condition is satisfied:*

$$\widetilde{\Phi} \equiv k\widetilde{k}\Psi\widetilde{\Psi} \operatorname{div} X + k\Psi \frac{d\widetilde{\Psi}}{dt} + \widetilde{k}\widetilde{\Psi} \frac{d\Psi}{dt} > 0 \ \ (< 0). \tag{4}$$

Let $W_0 = W \cup \widetilde{W}$, where

$$W = \big\{(x,y) \in \Omega : \ \Psi(x,y) = 0\big\}, \ \ \widetilde{W} = \big\{(x,y) \in \Omega : \ \widetilde{\Psi}(x,y) = 0\big\},$$

then the following assertions hold in the domain Ω: the set W_0 contains no equilibrium points of system (1)*; any trajectory of system* (1) *that encounters the set W_0 intersects it transversally; the set W_0 defines a curve with disjoint branches; and the limit cycles of system* (1) *that belong entirely to the domain Ω do not intersect the set W_0.*

Since the curves of the set W_0 divide the domain Ω in subdomains Ω_i in each of which B is a Dulac function in the classical sense, we find [3, p. 336] that the following assertion applies when evaluating the number of cycles of system (1) and localizing these cycles.

Theorem 3. *Suppose that in a connected domain Ω system* (1) *has the unique anti-saddle equilibrium point O and possesses a function B of the form* (3) *that satisfies condition* (4) *for $k < 0$, $\widetilde{k} < 0$. If sets W and $\widetilde{W}$ in the domain Ω consist of, respectively, s and $\widetilde{s}$ mutually embedded ovals that surround O, then in each of the $s + \widetilde{s} - 1$ ring-shaped subdomains Ω_i that are bounded by neighboring ovals ω_i and ω_{i+1} of the set W_0, system* (1) *has exactly one limit cycle, which is stable* (*unstable*) *for $\widetilde{\Phi}/(k\widetilde{k}\Psi\widetilde{\Psi}) < 0$ (> 0). System* (1) *can have at most $s + \widetilde{s}$ limit cycles in the domain Ω in total.*

However, none of the above theorems provides an exact estimate for the number of limit cycles of the considered systems (1), since to establish the existence or absence of a limit cycle in the external doubly connected subdomain Ω_s or $\Omega_{s+\widetilde{s}}$, one needs to conduct additional research and examine the influence of the other equilibrium points of rest or construct an additional transversal closed curve that embraces an external oval that corresponds to the function $B = |\Psi|^{\frac{1}{k}}$ or a function B of the form (3).

Now we will present our approaches to establishing the exact number of limit cycles of system (1) in the domain Ω, the approaches being based on constructing a closed transversal curve that surrounds the external oval of the function B in a doubly connected subdomain Ω_s with the use of an additional application of the Dulac or Dulac–Cherkas criterion. The gist of the first approach is expressed by the following assertion.

Theorem 4. *Suppose that the assumptions of Theorem* 1 *are valid, and system* (1) *has a second Dulac–Cherkas function* $\widetilde{\Psi}(x,y)$ *for* $\widetilde{k} < 0$ *in the domain* Ω *such that the set* $\widetilde{W}$ *consists of* $s+1$ *ovals in* Ω *that surround the point* O*. Then system* (1) *has exactly* s *limit cycles in the domain* Ω*.*

Proof. By virtue of Theorem 1, the existence of a Dulac–Cherkas function $\Psi(x,y)$ that defines s ovals in the domain Ω implies the existence of $s-1$ limit cycles of system (1) in the ring-shaped domains Ω_i, $i = 1, \dots, s-1$, bounded by neighboring ovals ω_i and ω_{i+1} and admits the existence of one limit cycle in the doubly connected subdomain Ω_s. By virtue of Theorem 1, the existence of the second Dulac–Cherkas function $\widetilde{\Psi}(x,y)$, that defines $s+1$ ovals in the domain Ω implies the existence of s limit cycles of system (1) in the ring-shaped domains $\widetilde{\Omega}_i$, $i = 1, \dots, s$, bounded by neighboring ovals of the set $\widetilde{W}$ and admits the existence of one limit cycle in the doubly connected subdomain $\widetilde{\Omega}_{s+1}$, that lies in between the external oval of the set $\widetilde{W}$ and the boundary $\partial\Omega$ of the domain Ω. The simultaneous existence of the functions Ψ and $\widetilde{\Psi}$ guarantees the existence of one limit cycle in the subdomain $\Omega_s \setminus \widetilde{\Omega}_{s+1}$ and rules out the existence of a limit cycle in the subdomain $\widetilde{\Omega}_{s+1}$. Hence it follows that system (1) has exactly s limit cycles in the domain Ω. It completes the proof of the theorem. □

A second approach can be described as follows.

Theorem 5. *Suppose that the assumptions of Theorem* 1 *hold, and, in addition, that in the domain* Ω *system* (1) *has a Dulac function* B *of the form* (3) *that satisfies the assumptions of Theorem* 3*, with the set* $\widetilde{W}$ *consisting of a single oval that is situated in the doubly connected domain* Ω_s *and surrounds all the ovals of the set* W*. Then system* (1) *has exactly* s *limit cycles in the domain* Ω*.*

Proof. The existence of $s-1$ limit cycles of system (1) in the case where the Dulac–Cherkas function $\Psi(x,y)$ exists can be proved similarly to Theorem 4. The existence of one more limit cycle in the doubly connected subdomain $\widetilde{\Omega}_s \subset \Omega_s$ in between the external oval of the set W and the single oval of the set $\widetilde{W}$ follows from Theorem 3. The simultaneous existence of the function Ψ and a function B of the form (3) guarantees that system (1) has exactly s limit cycles in the domain Ω. The proof of the theorem is complete. □

If the usage of Theorem 5 does not enable the construction of a function $\widetilde{\Psi}$, that satisfies inequality (3), one can relinquish the sign-definiteness of the function $\widetilde{\Phi}$ and use the condition of transversality of the set

$$V = \big\{(x,y) \in \Omega : \ \widetilde{\Phi} = 0\big\}$$

with respect to the vector field X of system (1). This constitutes the essence of the third approach.

Theorem 6. *Suppose that the assumptions of Theorem* 1 *are valid and there exists such a function* $\widetilde{\Psi}(x,y) \in C^1(\Omega)$ *with* $\widetilde{k} < 0$ *that in the domain* Ω *the set* $\widetilde{W}$ *intersects neither the set* V *nor the set* W*. Then the set* $\widetilde{W}$ *is transversal to the vector field* X *and is disjoint with the limit cycles of system* (1) *that belong entirely to the domain* Ω*.*

Proof. We consider the set $\widetilde{W}$. Since $\widetilde{W}$ and V are disjoint sets, it follows that the condition $\widetilde{\Phi} > 0(< 0)$ is satisfied on the set $\widetilde{W}$. By virtue of inequality (4), the condition $k\Psi\frac{d\widetilde{\Psi}}{dt} > 0$ (< 0) is satisfied on the curve $\widetilde{\Psi} = 0$ along any solution of system (1). Since the set $\widetilde{W}$ does not intersect the set W, it follows from the above inequality that the condition $\frac{d\widetilde{\Psi}}{dt} > 0$ (< 0) is satisfied. Consequently, any trajectory of system (1) intersects the curve $\widetilde{\Psi} = 0$ transversally.

Without loss of generality, we consider the case $\frac{d\widetilde{\Psi}}{dt} > 0$. Let us show that the limit cycles cannot intersect the curve $\widetilde{\Psi} = 0$. Suppose the contrary is true, then a point on the limit cycle can get with time onto the curve $\widetilde{\Psi} = 0$ only from a set in which $\widetilde{\Psi} < 0$ and should necessarily leave into a set in which $\widetilde{\Psi} > 0$. However, when moving along the limit cycle, the point should return into the original position in the domain $\widetilde{\Psi} = 0$, which is impossible in view of the inequality? $\frac{d\widetilde{\Psi}}{dt} > 0$. The obtained contradiction implies that the limit cycles of system (1) cannot intersect the curve $\widetilde{\Psi} = 0$ and it completes the proof. □

Remark 3. Theorems 4–6 persist if the function $\widetilde{\Psi}$ is found not in the entire domain Ω but only in the domain Ω_s or in its doubly connected subdomain $G_s \subset \Omega_s$ surrounding the equilibrium point O.

Theorem 7. *Suppose that the assumptions of Theorem* 1 *are valid and system* (1) *has a closed transversal curve that lies in a doubly connected subdomain Ω_s that surrounds the external oval of the set W, two of them forming the boundary of a ring-shaped domain $\widetilde{\Omega}_s \subset \Omega_s$. Then, if the trajectories of system* (1) *enter, as t increases, the interior of the domain $\widetilde{\Omega}_s$ from outside* (*or vice versa*) *through the boundary $\partial\widetilde{\Omega}_s$, then there exists the unique stable* (*or unstable*) *limit cycle of system* (1) *in the subdomain $\widetilde{\Omega}_s$ and system* (1) *has exactly s limit cycles in the domain Ω in total.*

Proof. According to Theorem 4, the existence of a Dulac–Cherkas function $\Psi(x, y)$ ensures the existence of $s - 1$ limit cycles of system (1) encircled by the external oval ω_s of the set W. In accordance with the Dulac criterion, system (1) can have no more than one limit cycle in the doubly connected subdomain Ω_s. On the other hand, if the trajectories of system (1) enter, as t increases, the interior of the subdomain $\widetilde{\Omega}_s$ from outside (or vice versa) through the boundary $\partial\widetilde{\Omega}_s$, then, according to the Poincare theorem [3, p. 64], there exists at least one stable (or unstable) limit cycle in the subdomain $\widetilde{\Omega}_s$. Thus, we establish the uniqueness of the limit cycle in $\widetilde{\Omega}_s$. Consequently, system (1) has exactly s limit cycles in the domain Ω. The proof is complete. □

A detailed presentation of the approaches developed by us and their application to some classes of systems (1) are contained in our paper [5]. Our paper [6] also shows that these approaches can be effectively implemented to establish the exact number of limit cycles surrounding several equilibrium points of systems (1), the total Poincaré index of which is +1.

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