Stability Analysis of Invariant Tori of Nonlinear Extensions of Dynamical Systems on Torus Using Quadratic Forms

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1 Introduction and preliminaries

A set of fundamental results of the mathematical theory of multifrequency oscillations have been developed by A. M. Samoilenko and summarized in [11]. In particular, these studies include the problems of the existence and stability of invariant manifolds of dynamical systems defined in the direct product of m-dimensional torus $\mathcal{T}_m$ and n-dimensional Euclidean space $\mathbb{R}^n$. In [5], the stability properties of invariant tori have been studied in terms of sign-definite quadratic forms. In this paper, we establish less restrictive (compared to [5]) conditions for exponential stability and instability of the trivial invariant torus of nonlinear extension of dynamical system on torus which are formulated in terms of quadratic forms that are sign-definite in nonwandering set Ω of dynamical system on torus and allowed to be sign-indefinite in $\mathcal{T}_m \setminus \Omega$. For further details we refer a reader to the extended version of this contribution [2]. The corresponding results for linear extensions of dynamical systems on torus have been obtained in [1, 3, 7–10].

We consider the following system defined in $\mathcal{T}_m \times \mathbb{R}^n$

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = P(\varphi, x)x, \tag{1.1}$$

where $\varphi = (\varphi_1, \dots, \varphi_m)^\top \in \mathcal{T}_m$, $x = (x_1, \dots, x_n)^\top \in \mathbb{R}^n$, function P is continuous in $\mathcal{T}_m \times \mathbb{R}^n$ and for every $x \in \mathbb{R}^n$ $P(\,\cdot\,, x), a(\,\cdot\,) \in C(\mathcal{T}_m)$; $C(\mathcal{T}_m)$ is a space of continuous 2π-periodic with respect to each of the components φ_v, $v = 1, \dots, m$ functions defined on $\mathcal{T}_m$. We assume that the following conditions hold:

$$\exists\, M > 0 \ \text{ such that } \ \forall\, (\varphi, x) \in \mathcal{T}_m \times \mathbb{R}^n \quad \|P(\varphi, x)\| \le M; \tag{1.2}$$

$$\forall\, r > 0 \ \ \exists\, L = L(r) > 0 \ \text{ such that } \ \forall\, x', x'', \ \ \|x'\| \le r, \ \ \|x''\| \le r, \ \ \forall\, \varphi \in \mathcal{T}_m$$
$$\big\|P(\varphi, x'') - P(\varphi, x')\big\| \le L\|x'' - x'\|; \tag{1.3}$$

$$\exists\, A > 0 \ \ \forall\, \varphi', \varphi'' \in \mathcal{T}_m \ \ \|a(\varphi'') - a(\varphi')\| \le A\|\varphi'' - \varphi'\|. \tag{1.4}$$

Condition (1.4) guarantees that the system

$$\frac{d\varphi}{dt} = a(\varphi) \tag{1.5}$$

generates a dynamical system on $\mathcal{T}_m$, which will be denoted by $\varphi_t(\varphi)$.

Definition 1.1 ([6])**.** A point $\varphi \in \mathcal{T}_m$ is called a nonwandering point of dynamical system (1.5) if there exist a neighbourhood $U(\varphi)$ and a moment of time $T = T(\varphi) > 0$ such that

$$U(\varphi) \cap \varphi_t(U(\varphi)) = \varnothing \ \ \forall\, t \geq T.$$

Let us denote by Ω a set of all nonwandering points of (1.5). Since $\mathcal{T}_m$ is a compact set, the set Ω is nonempty, invariant, and compact subset of $\mathcal{T}_m$ [11]. Additionally, the following holds:

Lemma 1.1 ([6])**.** *For any $\varepsilon > 0$ there exist $T(\varepsilon) > 0$ and $N(\varepsilon) > 0$ such that for any $\varphi \notin \Omega$ the corresponding trajectory $\varphi_t(\varphi)$ spends only a finite time that is bounded by $T(\varepsilon)$ outside the ε-neighbourhood of the set Ω, and leaves this set not more than $N(\varepsilon)$ times.*

Definition 1.2 ([11])**.** Trivial invariant torus $x = 0$, $\varphi \in \mathcal{T}_m$ of the system (1.1) is called exponentially stable if there exist constants $K > 0$, $\gamma > 0$, and $\delta > 0$ such that for all $\varphi \in \mathcal{T}_m$ and for all $x^0 \in \mathbb{R}^n$, $\|x^0\| \leq \delta$ it holds that

$$\forall\, t \geq 0 \ \ \|x(t, \varphi, x^0)\| \leq K\|x^0\|e^{-\gamma t},$$

where $x(t, \varphi, x^0)$ is a solution to the Cauchy problem

$$\frac{dx}{dt} = P(\varphi_t(\varphi), x)x, \quad x(0) = x^0.$$

In [4], the conditions for the exponential stability of the trivial invariant torus of the system (1.1) have been established in terms of the properties of function $\varphi \mapsto P(\varphi, 0)$ in the nonwandering set Ω of dynamical system (1.5):

Lemma 1.2 ([4])**.** *Let*

$$\forall\, \varphi \in \Omega \ \ \lambda(\varphi, 0) < 0, \tag{1.6}$$

where $\lambda(\varphi, x)$ is the largest eigenvalue of the matrix $\widehat{P}(\varphi, x) = \frac{1}{2}\left(P(\varphi, x) + P^T(\varphi, x)\right)$. Then the trivial invariant torus of system (1.1) *is exponentially stable.*

The following example demonstrates the case when the trivial invariant torus is exponentially stable (this will be proven in Theorem 2.1), however the condition (1.6) does not hold.

Example 1.1. Consider a system defined in $\mathcal{T}_1 \times \mathbb{R}^2$

$$\frac{d\varphi}{dt} = -\sin^2\Big(\frac{\varphi}{2}\Big), \quad \begin{pmatrix} \dfrac{dx_1}{dt} \\[2mm] \dfrac{dx_2}{dt} \end{pmatrix} = \begin{pmatrix} \sin(\varphi + x_1 + x_2)x_1 & -x_2 \\ x_1 & -\sin(x_1 - x_2 - \varphi)x_2 \end{pmatrix}. \tag{1.7}$$

Dynamical system on torus $\mathcal{T}_1$ that are generated by (1.7) has a nonwandering set $\Omega = \{\varphi = 0\}$. However, the matrix $\widehat{P}(0, \overline{0}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ does not satisfy condition (1.6).

In the following section, we prove new sufficient conditions that allow concluding exponential stability of trivial invariant torus of system (1.7).

2 Main results

For any $\varphi \in \mathcal{T}_m$, $x \in \mathbb{R}^n$ let us denote

$$\widehat{S}(\varphi,x) = \frac{\partial S(\varphi,x)}{\partial \varphi}\, a(\varphi) + \frac{\partial S(\varphi,x)}{\partial x}\, (P(\varphi,x)x) + S(\varphi,x)P(\varphi,x) + P^T(\varphi,x)S(\varphi,x), \tag{2.1}$$

where $S = S(\varphi,x)$ is a symmetric matrix of a class $C^1(\mathcal{T}_m \times \mathbb{R}^n)$.

Theorem 2.1. *Let there exist a symmetric matrix $S = S(\varphi,x) \in C^1(\mathcal{T}_m \times \mathbb{R}^n)$ such that*

$$\forall\, \varphi \in \Omega \ \ S(\varphi,0) > 0, \ \ \widehat{S}(\varphi,0) < 0.$$

Then the trivial torus of system (1.1) *is exponentially stable.*

Example 2.1 (revisited)**.** Let us illustrate the usage of Theorem 2.1 for system (1.7). Let $S = S(\varphi,x) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} > 0$. Then, $\widehat{S}(0,\overline{0}) = \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix} < 0$ which guarantees the exponential stability of the trivial invariant torus.

The following theorem provides sufficient conditions for instability of the trivial torus of system (1.1) in terms of sign-definite on the set Ω quadratic forms.

Theorem 2.2. *Let there exist a symmetric matrix $S = S(\varphi,x)$ of the class $C^1(\mathcal{T}_m \times \mathbb{R}^n)$ such that for the matrix* (2.1) *and for the quadratic form $V(\varphi,x) = (S(\varphi,x)x,x)$ the following conditions hold:*

$$\begin{gathered} \forall\, \varphi \in \Omega \ \ \widehat{S}(\varphi,0) > 0, \\ \forall\, \delta > 0 \ \ \exists\, x_0 \in \mathbb{R}^n, \ \ \|x_0\| < \delta, \ \ \exists\, \varphi_0 \in \Omega \ \ \textit{such that} \ \ V(\varphi_0,x_0) > 0. \end{gathered}$$

Then the trivial torus of system (1.1) *is unstable.*

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