On the Problem on Minimization of the Functional Generated by a Sturm–Liouville Problem

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1 Introduction

Consider the Sturm–Liouville problem

$$y''+Q(x)y+\lambda y=0,\ \ x\in(0,1), \tag{1.1}$$

$$y(0)=y(1)=0, \tag{1.2}$$

where Q belongs to the set $T_{\alpha,\beta,\gamma}$ of all real-valued locally integrable on $(0,1)$ functions with non-negative values such that the following integral condition holds

$$\int_0^1 x^{\alpha}(1-x)^{\beta}Q^{\gamma}(x)\,dx=1,\ \ \alpha,\beta,\gamma\in\mathbb{R},\ \ \gamma\neq 0. \tag{1.3}$$

A function y is a *solution to problem* (1.1), (1.2) if it is absolutely continuous on the segment $[0,1]$, satisfies (1.2), its derivative y' is absolutely continuous on any segment $[\rho,1-\rho]$, where $0<\rho<\frac{1}{2}$, and equality (1.1) holds almost everywhere in the interval $(0,1)$.

For any function $Q\in T_{\alpha,\beta,\gamma}$ by H_Q we denote the closure of the set $C_0^{\infty}(0,1)$ with respect to the norm

$$\|y\|_{H_Q}=\bigg(\int_0^1 y'^2dx+\int_0^1 Q(x)y^2\,dx\bigg)^{\frac{1}{2}}.$$

We consider the functional generated by problem (1.1), (1.2)

$$R[Q,y]=\frac{\int_0^1 y'^2\,dx-\int_0^1 Q(x)y^2\,dx}{\int_0^1 y^2\,dx}\,.$$

We give estimates for

$$m_{\alpha,\beta,\gamma}=\inf_{Q\in T_{\alpha,\beta,\gamma}}\ \inf_{y\in H_Q\setminus\{0\}}R[Q,y],\quad M_{\alpha,\beta,\gamma}=\sup_{Q\in T_{\alpha,\beta,\gamma}}\ \inf_{y\in H_Q\setminus\{0\}}R[Q,y].$$

Remark 1.1. This work is the continuation of the study of estimates for the first eigenvalue of Sturm–Liouville problems with integral conditions on the potential, which was initiated by Y. V. Egorov and V. A. Kondratiev [1]. The history of the research can be found in [2].

2 Main results

2.1 On precise estimates for $M_{\alpha,\beta,\gamma}$ as $\gamma < -1$, $\alpha, \beta > 2\gamma - 1$

It is proved [3] that $M_{\alpha,\beta,\gamma} \leqslant \pi^2$ for all α, β, γ, $\gamma \neq 0$, and $M_{\alpha,\beta,\gamma} < \pi^2$ as $\gamma < 0$, $\alpha, \beta > 3\gamma - 1$.

In case of $\gamma < 0$, using the Hölder inequality for any functions $Q \in T_{\alpha,\beta,\gamma}$ and $y \in H_Q$, we obtain

$$\int\limits_0^1 Q(x)y^2\,dx \geqslant \bigg(\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx\bigg)^{\frac{\gamma-1}{\gamma}}$$

and

$$\inf_{y\in H_Q\setminus\{0\}} R[Q,y] \leqslant \inf_{y\in H_Q\setminus\{0\}} G[y],$$

where

$$G[y] = \frac{\int\limits_0^1 y'^2\,dx - \big(\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx\big)^{\frac{\gamma-1}{\gamma}}}{\int\limits_0^1 y^2\,dx}\,.$$

Consider the functional G in $H_0^1(0,1)$. It is proved [4] that for $\gamma < 0$, $\alpha, \beta > 2\gamma - 1$ the functional G is bounded below in $H_0^1(0,1)$ and there exists

$$m = \inf_{y\in H_0^1(0,1)\setminus\{0\}} G[y].$$

Similarly to [4] we prove that for $\gamma < 0$, $\alpha, \beta > 2\gamma - 1$ any minimizing sequence of the functional G in $H_0^1(0,1)$ converges to some function $u \in H_0^1(0,1)$ and

$$\inf_{y\in H_0^1(0,1)\setminus\{0\}} G[y] = G[u] = m.$$

As in the case of $\alpha = \beta = 0$ [2] we prove that function u is positive on $(0,1)$.

For $0 < \varepsilon < \frac{1}{3}$, we consider the function

$$v(x) = \begin{cases} 0, & x \in [0,\varepsilon] \cup [1-\varepsilon, 1],\\ u, & x \in (\varepsilon, 1-\varepsilon) \end{cases}$$

and its averaging v_ρ with $\rho = \frac{\varepsilon}{2}$ (see, for example, [5, I, § 1]). Then for any function $Q \in T_{\alpha,\beta,\gamma}$, we obtain

$$\inf_{y\in H_Q\setminus\{0\}} R[Q,y] \leqslant \inf_{y\in H_Q\setminus\{0\}} G[y] \leqslant \lim_{\rho\to 0} G[v_\rho] = G[u] = m$$

and

$$M_{\alpha,\beta,\gamma} = \sup_{Q\in T_{\alpha,\beta,\gamma}} \inf_{y\in H_Q\setminus\{0\}} R[Q,y] \leqslant m.$$

On $(0,1)$ we consider the function $Q_*(x) = x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{2}{\gamma-1}}$ which satisfies the integral condition (1.3) and $u \in H_{Q_*}$. Since the function u is the first eigenfunction for problem (1.1)–(1.3) for $Q = Q_*$ and the first eigenvalue $\lambda_1(Q_*) = m$, then

$$\inf_{y\in H_{Q*}\setminus\{0\}} R[Q_*,y] = R[Q_*,u] = m.$$

Therefore, $M_{\alpha,\beta,\gamma} \geqslant m$. Hence, the following theorem holds.

Theorem 2.1. *If $\gamma < -1$, $\alpha, \beta > 2\gamma - 1$ and $m = \inf\limits_{y \in H_0^1(0,1) \setminus \{0\}} G[y]$, then there exist functions $Q_* \in T_{\alpha,\beta,\gamma}$ and $u \in H_{Q_*}$, $u > 0$ on $(0,1)$, such that $M_{\alpha,\beta,\gamma} = R[Q_*, u]$, moreover, u satisfies the equation*

$$u'' + mu = -x^{\frac{\alpha}{1-\gamma}} (1-x)^{\frac{\beta}{1-\gamma}} u^{\frac{\gamma+1}{\gamma-1}} \tag{2.1}$$

and the integral condition

$$\int_0^1 x^{\frac{\alpha}{1-\gamma}} (1-x)^{\frac{\beta}{1-\gamma}} u^{\frac{2\gamma}{\gamma-1}} \, dx = 1. \tag{2.2}$$

2.2 On estimates for $M_{\alpha,\beta,\gamma}$ as $\gamma > 0$

Theorem 2.2.

- *If $\gamma > 1$, then $M_{\alpha,\beta,\gamma} = \pi^2$.*
- *If $0 < \gamma \leqslant 1$, $\alpha \leqslant 2\gamma - 1$, $-\infty < \beta < +\infty$ or $\beta \leqslant 2\gamma - 1$, $-\infty < \alpha < +\infty$, then $M_{\alpha,\beta,\gamma} = \pi^2$.*
- *If $0 < \gamma < 1$, $\alpha, \beta > 3\gamma - 1$, then $M_{\alpha,\beta,\gamma} < \pi^2$.*
- *If $0 < \gamma < 1/2$, $\alpha, \beta \geqslant 0$, then $M_{\alpha,\beta,\gamma} < \pi^2$.*
- *If $1/2 \leqslant \gamma < 1$, $2\gamma - 1 < \alpha, \beta \leqslant 3\gamma - 1$, then $M_{\alpha,\beta,\gamma} < \pi^2$.*

Remark 2.1. The result $M_{0,0,\gamma} < \pi^2$ as $0 < \gamma < 1/2$ was obtained in [6].

Remark 2.2. We can give some lower bounds for $M_{\alpha,\beta,\gamma}$ in cases of $\gamma < 0$ or $0 < \gamma < 1$:

- If $\gamma < 0$, $\alpha, \beta \geqslant 0$, then $M_{\alpha,\beta,\gamma} \geqslant \pi^2 - 1$.
- If $\gamma < 0$, $2\gamma - 1 < \alpha < 0 \leqslant \beta$, then $M_{\alpha,\beta,\gamma} \geqslant (1 - 4(\alpha - 2\gamma + 1)^{\frac{1}{\gamma}})\pi^2$.
- If $\gamma < 0$, $2\gamma - 1 < \beta < 0 \leqslant \alpha$, then $M_{\alpha,\beta,\gamma} \geqslant (1 - 4(\beta - 2\gamma + 1)^{\frac{1}{\gamma}})\pi^2$.
- If $\gamma < 0$, $2\gamma - 1 < \alpha, \beta < 0$, then $M_{\alpha,\beta,\gamma} \geqslant (1 + \theta^{\frac{1}{\gamma}} \cdot 2^{\frac{\theta + 4\gamma - 2}{\gamma}})\pi^2$, $\theta = \min\{\alpha, \beta\} - 2\gamma + 1$.

2.3 Some estimates for $m_{\alpha,\beta,\gamma}$ below

Theorem 2.3.

- *If $\gamma < 0$ or $0 < \gamma < 1$, then $m_{\alpha,\beta,\gamma} = -\infty$.*
- *If $\gamma = 1$ and $\alpha, \beta \leqslant 0$, then $m_{\alpha,\beta,\gamma} \geqslant \frac{3}{4}\pi^2$.*
- *If $\gamma = 1$, $\beta \leqslant 0 < \alpha \leqslant 1$ or $\alpha \leqslant 0 < \beta \leqslant 1$, then $m_{\alpha,\beta,\gamma} \geqslant 0$.*
- *If $\gamma = 1$, $0 < \alpha, \beta \leqslant 1$, then $-\pi^2 \leqslant m_{\alpha,\beta,\gamma} \leqslant \pi^2$.*
- *If $\gamma > 1$ and $0 < \alpha$, $\beta \leqslant 2\gamma - 1$, then*

$$m_{\alpha,\beta,\gamma} \geqslant \Big(1 - 2^{\frac{3\gamma-2}{\gamma}} \Big(\frac{2\gamma - 1}{\gamma}\Big)^{\frac{2\gamma-1}{\gamma}}\Big)\pi^2.$$

- *If* $\gamma > 1$ *and* $\beta \leqslant 0 < \alpha \leqslant 2\gamma - 1$ *or* $\alpha \leqslant 0 < \beta \leqslant 2\gamma - 1$*, then*

$$m_{\alpha,\beta,\gamma} \geqslant \Big(1 - \Big(\frac{2\gamma - 1}{\gamma}\Big)^{\frac{2\gamma-1}{\gamma}}\Big)\pi^2.$$

- *If* $\gamma > 1$ *and* $\alpha, \beta \leqslant 0$*, then* $m_{\alpha,\beta,\gamma} \geqslant 0$*.*

Theorem 2.4. *If* $\gamma > 1$ *and* $\alpha, \beta < 2\gamma - 1$*, then there exist functions* $Q_* \in T_{\alpha,\beta,\gamma}$ *and* $u \in H_{Q_*}$*,* $u > 0$ *on* $(0, 1)$*, such that* $m_{\alpha,\beta,\gamma} = R[Q_*, u] = m$*, moreover,* u *satisfies equation* (2.1) *and the integral condition* (2.2)*.*

References

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