Asymptotic Behaviour of Solutions of Third-Order Differential Equations with Rapid Varying Nonlinearities

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We consider the differential equation

$$y''' = \alpha_0 p(t)\varphi(y), \tag{1}$$

where $\alpha_0 \in \{-1,1\}$, $p : [a,\omega[\,\to\,]0,+\infty[$ is a continuous function, $-\infty < a < \omega \le +\infty$, $\varphi : \Delta_{Y_0} \to\,]0,+\infty[$ is a twice continuously differentiable function such that

$$\varphi'(y) \neq 0 \ \text{ for } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \varphi(y) = \begin{cases} \text{or} & 0,\\ \text{or} & +\infty, \end{cases} \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\varphi(y)\varphi''(y)}{\varphi'^2(y)} = 1, \tag{2}$$

Y_0 equals either zero or $\pm\infty$, Δ_{Y_0} – some one-sided neighborhood of Y_0.

From identity

$$\frac{\varphi''(y)\varphi(y)}{\varphi'^2(y)} = \frac{(\frac{\varphi'(y)}{\varphi(y)})'}{(\frac{\varphi'(y)}{\varphi(y)})^2} + 1 \ \text{ for } \ y \in \Delta_{Y_0}$$

and conditions (2) it follows that

$$\frac{\varphi'(y)}{\varphi(y)} \sim \frac{\varphi''(y)}{\varphi'(y)}, \ \ y \to Y_0 \ \ (y \in \Delta_{Y_0}), \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{y\varphi'(y)}{\varphi(y)} = \pm\infty.$$

It means that in the considered equation the continuous function φ and its first order derivatives are [5, Chapter 3, Section 3.4, Lemmas 3.2, 3.3, pp. 91–92] rapidly change at $y \to Y_0$.

For two-term differential equations of the second order with nonlinearities satisfying condition (2), the asymptotic properties of solutions were studied in the works of M. Maric [5], V. M. Evtukhov and his students N. G. Drik, V. M. Kharkov, A. G. Chernikova [1–3].

In the works of V. M. Evtukhov, A. G. Chernikova [1] for the differential equation (1) of the second order in the case when φ satisfies condition (2), the asymptotic properties of so-called $P_\omega(Y_0,\lambda_0)$-solutions were studied with $\lambda_0 \in \mathbb{R} \setminus \{0,1\}$. In this work, we propose the distribution of [1] results to third-order differential equations.

Solution y of the differential equation (1) specified on the interval $[t_0,\omega[\, \subset [a,\omega[$ calls $P_\omega(Y_0,\lambda_0)$-solution, if it satisfies the following conditions:

$$\lim_{t\uparrow\omega} y(t) = Y_0, \quad \lim_{t\uparrow\omega} y^{(k)}(t) = \begin{cases} \text{or} & 0,\\ \text{or} & \pm\infty, \end{cases} k = 1,2, \quad \lim_{t\uparrow\omega} \frac{y''^2(t)}{y'''(t)y'(t)} = \lambda_0.$$

The goal of this work is to establish the necessary and sufficient conditions for the existence for the equation (1) $P_\omega(Y_0,\lambda_0)$-solutions in the non-singular case, when $\lambda_0 \in \mathbb{R} \setminus \{0,1,\frac{1}{2}\}$, as well as asymptotic for $t \uparrow \omega$ representations for such solutions and their derivatives up to the second order inclusively.

Without loss of generality, we will further assume that

$$\Delta_{Y_0}=\begin{cases}[y_0,Y_0[\,, & \text{if } \Delta_{Y_0} \text{ is a left neighborhood of } Y_0,\\ ]Y_0,y_0], & \text{if } \Delta_{Y_0} \text{ is a right neighborhood of } Y_0,\end{cases} \tag{3}$$

where $y_0\in\mathbb{R}$ is such that $|y_0|<1$, when $Y_0=0$ and $y_0>1$ ($y_0<-1$), when $Y_0=+\infty$ (when $Y_0=-\infty$).

A function $\varphi:\Delta_{Y_0}\to\mathbb{R}\setminus\{0\}$, satisfying condition (2), belongs to the class $\Gamma_{Y_0}(Z_0)$, that was introduced in the work [1], which extends the class of function Γ, introduced by L. Khan (see, for example, [4, Chapter 3, Section 3.10, p. 175]). Using properties from this class, main results below are obtained.

We introduce the necessary auxiliary notation. We assume that the domain of the function $\varphi\in\Gamma_{Y_0}(Z_0)$ is determined by formula (3). Next, we set

$$\mu_0=\operatorname{sgn}\varphi'(y),\quad \nu_0=\operatorname{sgn}y_0,\quad \nu_1=\begin{cases}1, & \text{if } \Delta_{Y_0}=[y_0,Y_0[\,,\\ -1, & \text{if } \Delta_{Y_0}=]Y_0,y_0],\end{cases}$$

and introduce the following functions

$$\pi_\omega(t)=\begin{cases}t, & \text{if } \omega=+\infty,\\ t-\omega, & \text{if } \omega<+\infty,\end{cases}\quad J(t)=\int\limits_A^t\pi_\omega^2(\tau)p(\tau)\,d\tau,\quad \Phi(y)=\int\limits_B^y\frac{ds}{\varphi(s)}\,,$$

where

$$A=\begin{cases}\omega, & \text{if } \displaystyle\int\limits_a^\omega\pi_\omega^2(\tau)p(\tau)\,d\tau=const,\\ a, & \text{if } \displaystyle\int\limits_a^\omega\pi_\omega^2(\tau)p(\tau)\,d\tau=\pm\infty,\end{cases}\qquad B=\begin{cases}Y_0, & \text{if } \displaystyle\int\limits_{y_0}^{Y_0}\frac{ds}{\varphi(s)}=const,\\ y_0, & \text{if } \displaystyle\int\limits_{y_0}^{Y_0}\frac{ds}{\varphi(s)}=\pm\infty.\end{cases}$$

Considering the definition of $P_\omega(Y_0,\lambda_0)$-solutions of the differential equation (1), we note that the numbers ν_0, ν_1 determine the signs of any $P_\omega(Y_0,\lambda_0)$-solution and of its first derivatives in some left neighborhood of ω. It is clear that the condition

$$\nu_0\nu_1<0 \ \text{ if } \ Y_0=0,\quad \nu_0\nu_1>0 \ \text{ if } \ Y_0=\pm\infty,$$

is necessary for the existence of such solutions.

Now we turn our attention to some properties of the function Φ. It retains a sign on the interval Δ_{Y_0}, tends either to zero or to $\pm\infty$, when $y\to Y_0$ and increasing by Δ_{Y_0}, because on this interval $\Phi'(y)=\frac{1}{\varphi(y)}>0$. Therefore, for it there is an inverse function $\Phi^{-1}:\Delta_{Z_0}\to\Delta_{Y_0}$, where due to the second of conditions (2) and the monotone increase of Φ^{-1},

$$Z_0=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\Phi(y)=\begin{cases}\text{or} & 0,\\ \text{or} & +\infty,\end{cases}\qquad \Delta_{Z_0}=\begin{cases}[z_0,Z_0[\,, & \text{or } \Delta_{Y_0}=[y_0,Y_0[\,,\\ ]Z_0,z_0], & \text{or } \Delta_{Y_0}=]Y_0,y_0],\end{cases}\qquad z_0=\varphi(y_0).$$

For $\lambda_0\in\mathbb{R}\setminus\{0;1;\frac12\}$ with using Φ^{-1} we also introduce the auxiliary functions

$$q(t)=\frac{\alpha_0(\lambda_0-1)^2\pi_\omega^3(t)p(t)\varphi\big(\Phi^{-1}(\alpha_0\frac{(\lambda_0-1)^2}{\lambda_0}(\lambda_0-1)J(t))\big)}{\lambda_0\Phi^{-1}\big(\alpha_0\frac{(\lambda_0-1)^2}{\lambda_0}J(t)\big)}\,,$$

$$H(t)=\frac{\Phi^{-1}(\alpha_0\frac{(\lambda_0-1)^2}{\lambda_0}J(t))\varphi'\big(\Phi^{-1}(\alpha_0\frac{(\lambda_0-1)^2}{\lambda_0}J(t))\big)}{\varphi(\Phi^{-1}(\alpha_0\frac{(\lambda_0-1)^2}{\lambda_0}J(t)))}\,.$$

For equation (1) the following assertions are valid.

Theorem 1. *Let $\lambda_0 \in \mathbb{R} \setminus \{0; 1; \frac{1}{2}\}$. Then for the existence for the differential equation* (1)*, $P_\omega(Y_0, \lambda_0)$-solutions, it is necessary to comply with the conditions*

$$\alpha_0 \nu_1 \lambda_0 > 0,$$
$$\nu_0 \nu_1 (2\lambda_0 - 1)(\lambda_0) \pi_\omega(t) > 0, \quad \alpha_0 \mu_0 \lambda_0 J(t) < 0 \ \text{ for } \ t \in (a, \omega),$$
$$\frac{\alpha_0}{\lambda_0} \lim_{t \uparrow \omega} J(t) = Z_0, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) J'(t)}{J(t)} = \pm\infty, \quad \lim_{t \uparrow \omega} q(t) = \frac{2\lambda_0 - 1}{\lambda_0 - 1}.$$

Moreover, for each such solution, the asymptotic representations

$$y(t) = \Phi^{-1}\Big(\alpha_0 \, \frac{(\lambda_0 - 1)^2}{\lambda_0} \, J(t)\Big) \Big[1 + \frac{o(1)}{H(t)}\Big] \ \text{ for } \ t \uparrow \omega,$$
$$y'(t) = \frac{(2\lambda_0 - 1)}{(\lambda_0 - 1)} \, \frac{\Phi^{-1}(\alpha_0 \frac{(\lambda_0 - 1)^2}{\lambda_0} J(t))}{\pi_\omega(t)} \, [1 + o(1)] \ \text{ for } \ t \uparrow \omega,$$
$$y''(t) = \frac{\lambda_0(2\lambda_0 - 1)}{(\lambda_0 - 1)^2} \, \frac{\Phi^{-1}(\alpha_0 \frac{(\lambda_0 - 1)^2}{\lambda_0} J(t))}{\pi_\omega^2(t)} \, [1 + o(1)] \ \text{ for } \ t \uparrow \omega$$

take place.

Theorem 2. *Let $\lambda_0 \in \mathbb{R} \setminus \{0; 1; \frac{1}{2}\}$, there exist a finite or equal to $\pm\infty$ limit*

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{(\frac{\varphi'(y)}{\varphi(y)})'}{(\frac{\varphi'(y)}{\varphi(y)})^2} \sqrt[3]{\Big(\frac{y \varphi'(y)}{\varphi(y)}\Big)^2}$$

and

$$\lim_{t \uparrow \omega} \Big[\frac{2\lambda_0 - 1}{\lambda_0 - 1} - q(t)\Big] |H(t)|^{\frac{2}{3}} = 0.$$

Then, the differential equation (1) *has at least one $P_\omega(Y_0, \lambda_0)$-solution which allows for $t \uparrow \omega$ the asymptotic representation*

$$y(t) = \Phi^{-1}\Big(\alpha_0 \frac{(\lambda_0 - 1)^2}{\lambda_0} \, J(t)\Big) \Big[1 + \frac{o(1)}{H(t)}\Big],$$
$$y'(t) = \frac{2\lambda_0 - 1}{(\lambda_0 - 1)\pi_\omega(t)} \, \Phi^{-1}\Big(\alpha_0 \, \frac{(\lambda_0 - 1)^2}{\lambda_0} \, J(t)\Big) \big[1 + o(1) H^{-\frac{2}{3}}\big],$$
$$y''(t) = \frac{\lambda_0(2\lambda_0 - 1)}{(\lambda_0 - 1)\pi_\omega^2(t)} \, \Phi^{-1}\Big(\alpha_0 \, \frac{(\lambda_0 - 1)^2}{\lambda_0} \, J(t)\Big) \big[1 + o(1) H^{-\frac{1}{3}}\big].$$

Moreover, in the case when $\mu_0 \lambda_0 (2\lambda_0 - 1)(\lambda_0 - 1) < 0$ there exists one-parameter family, but in the case $\mu_0 \lambda_0 (2\lambda_0 - 1)(\lambda_0 - 1) > 0$ there exists a two-parameter family.

References

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