Resonance Case of Full Separation of Countable Linear Homogeneous Differential System with Coefficients of Oscillating Type

V. V. Dzhashitova

Odessa I. I. Mechnikov National University, Odessa, Ukraine
E-mail: vera1685@yandex.ua

Let

$$G(\varepsilon_0) = \{t, \varepsilon:\ t \in \mathbf{R},\ \varepsilon \in [0, \varepsilon_0],\ \varepsilon_0 \in \mathbf{R}^+\}.$$

Definition 1. We say that the function $p(t,\varepsilon)$ belongs to the class $S(m;\varepsilon_0)$ $(m \in \mathbf{N} \cup \{0\})$ if

1) $p : G(\varepsilon_0) \to \mathbf{C}$;

2) $p(t,\varepsilon) \in C^m(G(\varepsilon_0))$ with respect to t;

3)
$$\frac{d^k p(t,\varepsilon)}{dt^k} = \varepsilon^k p_k^*(t,\varepsilon) \ \ (0 \le k \le m)$$
and
$$\|p\|_{S(m,\varepsilon_0)} \stackrel{def}{=} \sum_{k=0}^{m} \sup_{G(\varepsilon_0)} |p_k^*(t,\varepsilon)| < +\infty.$$

Definition 2. We say that the function $f(t,\varepsilon,\theta(t,\varepsilon))$ belongs to the class $F(m;\varepsilon_0;\theta)$ $(m \in \mathbf{N} \cup \{0\})$ if

$$f(t,\varepsilon,\theta(t,\varepsilon)) = \sum_{n=-\infty}^{\infty} f_n(t,\varepsilon) \exp(in\,\theta(t,\varepsilon)),$$

and

1) $f_n(t,\varepsilon) \in S(m,\varepsilon_0)$ $(n \in \mathbf{Z})$;

2)
$$\|f\|_{F(m;\varepsilon_0,\theta)} \stackrel{def}{=} \sum_{n=-\infty}^{\infty} \|f_n\|_{S(m;\varepsilon_0)} < +\infty;$$

3) $\theta(t,\varepsilon) = \int\limits_0^t \varphi(\tau,\varepsilon)\,d\tau$, $\varphi \in \mathbf{R}^+$, $\varphi \in S(m,\varepsilon_0)$, $\inf\limits_{G(\varepsilon_0)} \varphi(t,\varepsilon) = \varphi_0 > 0$.

Definition 3. We say that the infinite dimensional $x(t,\varepsilon) = \operatorname{col}(x_1(t,\varepsilon), x_2(t,\varepsilon), \dots)$ belongs to the class $S_1(m;\varepsilon_0)$ if $x_j \in S(m;\varepsilon_0)$ $(j = 1, 2, \dots)$, and

$$\|x\|_{S_1(m;\varepsilon_0)} \stackrel{def}{=} \sup_j \|x_j\|_{S(m;\varepsilon_0)} < +\infty.$$

Definition 4. We say that the infinite dimensional matrix $A(t,\varepsilon)=(a_{jk}(t,\varepsilon))_{j,k=1,2,\dots}$ belongs to the class $S_2(m;\varepsilon_0)$ if $a_{jk}\in S(m;\varepsilon_0)$, and

$$\|A\|_{S_2(m;\varepsilon_0)}\overset{def}{=}\sup_j\sum_{k=1}^{\infty}\|a_{jk}\|_{S(m;\varepsilon_0)}<+\infty.$$

Definition 5. We say that the infinite dimensional vector $x(t,\varepsilon,\theta)=\operatorname{col}(x_1(t,\varepsilon,\theta),x_2(t,\varepsilon,\theta),\dots)$ belongs to the class $F_1(m;\varepsilon_0,\theta)$ if $x_j\in F(m;\varepsilon_0;\theta)$ $(j=1,2,\dots)$, and

$$\|x\|_{F_1(m;\varepsilon_0,\theta)}\overset{def}{=}\sup_j\|x_j\|_{F(m;\varepsilon_0,\theta)}<+\infty.$$

Definition 6. We say that the infinite dimensional matrix $A(t,\varepsilon,\theta)=(a_{jk}(t,\varepsilon,\theta))_{j,k=1,2,\dots}$ belongs to the class $F_2(m;\varepsilon_0,\theta)$ if $a_{jk}\in F(m;\varepsilon_0,\theta)$, and

$$\|A\|_{F_2(m;\varepsilon_0,\theta)}\overset{def}{=}\sup_j\sum_{k=1}^{\infty}\|a_{jk}\|_{F(m;\varepsilon_0,\theta)}<+\infty.$$

Obviously, if $A\in F_2(m;\varepsilon_0;\theta)$, $x\in F_1(m;\varepsilon_0;\theta)$, then $Ax\in F_1(m;\varepsilon_0;\theta)$, and

$$\|Ax\|_{F_1(m;\varepsilon_0;\theta)}\le 2^m\|A\|_{F_2(m;\varepsilon_0;\theta)}\cdot\|x\|_{F_1(m;\varepsilon_0;\theta)}.$$

The condition $\|A\|_{F_2(m;\varepsilon_0;\theta)}<1$ guarantees the existence of a matrix

$$(E+A)^{-1}=E+\sum_{k=1}^{\infty}(-1)^kA^k,$$

where $E=\operatorname{diag}(1,1,\dots)$.

For any vector $x(t,\varepsilon,\theta)\in F_1(m;\varepsilon_0;\theta)$ we denote

$$\Gamma_n[x]=\frac{1}{2\pi}\int_0^{2\pi}x(t,\varepsilon,\theta)\exp(-in\theta)\,d\theta,\quad n\in\mathbf{Z}.$$

For infinite dimensional vectors $x=\operatorname{colon}(x_1,x_2,\dots)$, $y=\operatorname{colon}(y_1,y_2,\dots)$ we denote $[x,y]=\operatorname{colon}(x_1y_1,x_2y_2,\dots)$.

We consider the following countable system of differential equations

$$\frac{dx}{dt}=\Lambda(t,\varepsilon)x+\mu B^{(0)}(t,\varepsilon,\theta)x+\mu^2B(t,\varepsilon,\theta)x, \tag{1}$$

where

$$\begin{gathered}
t,\varepsilon\in G(\varepsilon_0),\quad x=\operatorname{colon}(x_1,x_2,\dots),\\
\Lambda(t,\varepsilon)=\operatorname{diag}\big[\lambda_1(t,\varepsilon),\lambda_2(t,\varepsilon),\dots\big]\in S_2(m;\varepsilon_0),\\
B^{(0)}(t,\varepsilon,\theta)=\operatorname{diag}\big[b_1(t,\varepsilon,\theta),b_2(t,\varepsilon,\theta),\dots\big]\in F_2(m;\varepsilon_0;\theta),\\
B(t,\varepsilon,\theta)=(b_{jk}(t,\varepsilon,\theta))_{j,k=1,2,\dots}\in F_2(m;\varepsilon_0;\theta),\\
b_{jj}(t,\varepsilon,\theta)\equiv 0\ \ (j=1,2,\dots),\quad \mu\in(0,\mu_0)\subset\mathbf{R}^+.
\end{gathered}$$

We suppose

$$\lambda_j(t,\varepsilon)-\lambda_k(t,\varepsilon)=in_{jk}\varphi(t,\varepsilon), \tag{2}$$

$n_{jk} \in \mathbf{Z}$ $(j,k=1,2,\dots)$, $\varphi(t,\varepsilon)$ – function in Definition 2. In this sense we say that we have a resonance case.

We study the problem on the existence of the transformation of kind

$$x=(E+Q(t,\varepsilon,\theta,\mu))y, \tag{3}$$

$y=\operatorname{colon}(y_1,y_2,\dots)$, $Q(t,\varepsilon,\theta,\mu)=(q_{jk}(t,\varepsilon,\theta,\mu))_{j,k=1,2,\dots} \in F_2(m_1;\varepsilon_2;\theta)$ $(m_1 \le m, \varepsilon_1 \le \varepsilon_0)$, $q_{jj}(t,\varepsilon,\theta,\mu)\equiv 0$, which leads the system (4) to kind:

$$\frac{dy}{dt}=D(t,\varepsilon,\theta,\mu)y, \tag{4}$$
$$D(t,\varepsilon,\theta,\mu)=\operatorname{diag}\big[d_1(t,\varepsilon,\theta,\mu),d_2(t,\varepsilon,\theta,\mu),\dots\big]\in F_2(m_1,\varepsilon_1;\theta).$$

We consider the auxiliary countable system of differential equations

$$\frac{dz}{dt}=i\varphi(t,\varepsilon)\Lambda_1 z+\mu U(t,\varepsilon,\theta)z+g(t,\varepsilon,\theta)+\mu^2C(t,\varepsilon,\theta)z+\mu^4[z,R(t,\varepsilon,\theta)z], \tag{5}$$

where

$$\begin{gathered}
t,\varepsilon\in G(\varepsilon_0),\ \ z=\operatorname{colon}(z_1,z_2,\dots),\ \ \Lambda_1=\operatorname{diag}[n_1,n_2,\dots],\ \ n_j\in\mathbf{Z}\ \ (j=1,2,\dots),\\
U=\operatorname{diag}\big[u_1(t,\varepsilon,\theta),u_2(t,\varepsilon,\theta),\dots\big]\in F_2(m;\varepsilon_0;\theta),\\
g=\operatorname{colon}\big(g_1(t,\varepsilon,\theta),g_2(t,\varepsilon,\theta),\dots\big)\in F_1(m;\varepsilon_0;\theta),\\
C=(c_{jk}(t,\varepsilon,\theta))_{j,k=1,2,\dots}\in F_2(m;\varepsilon_0;\theta),\ \ c_{jj}\equiv 0\ \ (j=1,2,\dots),\\
R\in F_2(m;\varepsilon_0;\theta),\ \ \mu\in(0,\mu_0)\subset\mathbf{R}^+.
\end{gathered}$$

Lemma 1. *Let the system* (5) *satisfy the next conditions:*

1) $\forall\, t,\varepsilon\in G(\varepsilon_0)$:
$$\int\limits_0^{2\pi} g_j(t,\varepsilon,\theta)\exp(-in_j\theta)\,d\theta=0,\ \ j=1,2,\dots;$$

2)
$$\inf_{G(\varepsilon_0)}\bigg|\int\limits_0^{2\pi} u_j(t,\varepsilon,\theta)\,d\theta\bigg|\ge\gamma>0,\ \ j=1,2,\dots.$$

Then there exists $\mu_1\in(0,\mu_0)$ such that $\forall\,\mu\in(0,\mu_1)$ and $\forall\, q\in\mathbf{N}$ there exists the transformation of kind

$$z=\sum_{s=0}^{2q-1}\xi^{(s)}(t,\varepsilon,\theta)\mu^s+\Phi(t,\varepsilon,\theta,\mu)z^{(1)}, \tag{6}$$

$\xi^{(s)}\in F_1(m;\varepsilon_0;\theta)$, $\Phi\in F_2(m;\varepsilon_0;\theta)$, *which leads the system* (6) *to kind:*

$$\begin{aligned}
\frac{dz^{(1)}}{dt}=\Big(\sum_{l=1}^{q}K^{(l)}(t,\varepsilon)\mu^l\Big)z^{(1)}+\varepsilon h^{(11)}(t,\varepsilon,\theta,\mu)+\mu^{2q}h^{(12)}(t,\varepsilon,\theta,\mu)\\
+\varepsilon V^{(1)}(t,\varepsilon,\theta,\mu)z^{(1)}+\mu^{q+1}P^{(1)}(t,\varepsilon,\theta,\mu)z^{(1)}\\
+\mu\big[R^{(11)}(t,\varepsilon,\theta,\mu)z^{(1)},R^{(12)}(t,\varepsilon,\theta,\mu)z^{(1)}\big],
\end{aligned}$$

where $K^{(l)}\in S_2(m;\varepsilon_0)$, and $\forall\,\mu\in(0,\mu_1)$; $h^{(11)},h^{(12)}\in F_1(m-1;\varepsilon_0;\theta)$, $V^{(1)},P^{(1)},R^{(11)},R^{(12)}\in F_2(m-1;\varepsilon_0;\theta)$.

We consider the countable linear homogeneous system of differential equations:

$$\frac{dx^{(0)}}{dt} = A(t,\varepsilon)x^{(0)}, \tag{7}$$

where $A(t,\varepsilon) \in S_2(m;\varepsilon_0)$.

Definition 7. The Green-matrix of the system (7) is the matrix $G(t,\tau,\varepsilon) = (g_{jk}(t,\tau,\varepsilon))_{j,k=1,2,\dots}$, such that

1) if $t \neq \tau$:
$$\frac{\partial G(t,\tau,\varepsilon)}{\partial t} = A(t,\varepsilon)G(t,\tau,\varepsilon), \quad \frac{\partial G(t,\tau,\varepsilon)}{\partial \tau} = -G(t,\tau,\varepsilon)A(\tau,\varepsilon);$$

2)
$$G(\tau+0,\tau,\varepsilon) - G(\tau-0,\tau,\varepsilon) = E, \quad G(t,t+0,\varepsilon) - G(t,t-0,\varepsilon) = -E.$$

If $t = \tau$, then Green-matrix is not defined.

Along with the system (7) consider the countable linear inhomogeneous system:

$$\frac{dx}{dt} = A(t,\varepsilon)x + f(t,\varepsilon,\theta), \tag{8}$$

where $f \in F_1(m;\varepsilon_0;\theta)$, matrix $A(t,\varepsilon)$ is the same as in the system (7).

Lemma 2. *Let the system* (7) *have the Green-matrix* $G(t,\tau,\varepsilon) = (g_{jk}(t,\tau,\varepsilon))_{j,k=1,2,\dots}$ *such that*

$$|g_{jk}(t,\tau,\varepsilon)| \leq M_0 \exp\big(-\gamma_0|t-\tau|\big),$$

where $M_0,\gamma_0 \in (0,+\infty)$, *and* M_0,γ_0 *do not depend on* t, τ, ε. *Then the system* (8) *has a unique particular solution* $x(t,\varepsilon,\theta) \in F_1(m;\varepsilon_0;\theta)$, *and there exists* $K_0 \in (0,+\infty)$ *such that*

$$\|x(t,\varepsilon,\theta)\|_{F_1(m;\varepsilon_0;\theta)} \leq \frac{K_0}{\gamma_0}\, \|f(t,\varepsilon,\theta)\|_{F_1(m;\varepsilon_0;\theta)}.$$

Lemma 3. *Let the system* (5) *be such that*

1) *the conditions of Lemma* 1 *hold;*

2) *for the linear homogeneous system*

$$\frac{dx}{dt} = \Big(\sum_{l=1}^{q} K^{(l)}(t,\varepsilon)\mu^l\Big)x,$$

where matrices $K^{(l)}(t,\varepsilon)$ *are defined by Lemma* 1, *there exists the Green-matrix* $G(t,\tau,\varepsilon,\mu) = (g_{jk}(t,\tau,\varepsilon,\mu))_{j,k=1,2,\dots}$ *such that*

$$|g_{jk}(t,\tau,\varepsilon,\mu)| \leq M_1 \exp\big(-\gamma_1\mu^{q_0}|t-\tau|\big),$$

$q_0 \in [1,q]$, $M_1,\gamma_1 \in (0,+\infty)$ *and do not depend on* t, τ, ε, μ.

Then there exist $\mu_2 \in (0,\mu_0)$, $\varepsilon_2(\mu) \in (0,\varepsilon_0)$ *such that* $\forall\, \mu \in (0,\mu_2)$, $\varepsilon \in (0,\varepsilon_2(\mu))$ *the system* (5) *has a particular solution, belonging to the class* $F_1(m-1;\varepsilon_2(\mu);\theta)$.

Now we return to the system (1) and make in it substitution (3). Taking into account the condition of diagonality of transformed system (4) and condition (2), we obtain the next countable system of differential equations for the elements q_{jk} $(j \neq k)$ of matrix Q:

$$\frac{dq_{jk}}{dt} = in_{jk}\varphi(t,\varepsilon)q_{jk} + \mu\big(b_j(t,\varepsilon,\theta) - b_k(t,\varepsilon,\theta)\big)q_{jk} + \mu^2 b_{jk}(t,\varepsilon,\theta) \\ + \mu^2 \sum_{\substack{s=1 \\ (s\neq j,\, s\neq k)}}^{\infty} b_{js}(t,\varepsilon,\theta)q_{sk} - \mu^2 q_{jk} \sum_{\substack{s=1 \\ (s\neq k)}}^{\infty} b_{ks}(t,\varepsilon,\theta)q_{sk}, \ \ j,k = 1,2,\dots; \ \ j \neq k. \tag{9}$$

The elements of the diagonal matrix D in system (4) are defined by formulas:

$$d_j(t,\varepsilon,\theta,\mu) = \lambda_j(t,\varepsilon) + \mu b_j(t,\varepsilon,\theta) + \mu \sum_{\substack{s=1 \\ (s\neq j)}}^{\infty} b_{js}(t,\varepsilon,\theta)q_{sj}(t,\varepsilon,\theta,\mu). \tag{10}$$

The substitution

$$q_{jk} = \mu^2 \widetilde{q}_{jk}, \ \ j,k = 1,2,\dots; \ \ j \neq k$$

leads the system (9) to kind:

$$\frac{d\widetilde{q}_{jk}}{dt} = in_{jk}\varphi(t,\varepsilon)\widetilde{q}_{jk} + \mu\big(b_j(t,\varepsilon,\theta) - b_k(t,\varepsilon,\theta)\big)\widetilde{q}_{jk} + b_{jk}(t,\varepsilon,\theta) \\ + \mu^2 \sum_{\substack{s=1 \\ (s\neq j,\, s\neq k)}}^{\infty} b_{js}(t,\varepsilon,\theta)\widetilde{q}_{sk} - \mu^4 \widetilde{q}_{jk} \sum_{\substack{s=1 \\ (s\neq k)}}^{\infty} b_{ks}(t,\varepsilon,\theta)\widetilde{q}_{sk}, \ \ j,k = 1,2,\dots; \ \ j \neq k. \tag{11}$$

In the system (11) index k is fixed, then for any $k = 1,2,\dots$ system (11) is the separate countable system of the differential equations for $\widetilde{q}_{1k}, \widetilde{q}_{2k}, \dots, \widetilde{q}_{k-1,k}, \widetilde{q}_{k+1,k}, \dots$. It is not difficult to see that vector-form of such system has a kind (5). Then we can prove the validity of the next theorem.

Theorem. *Let for the system* (1) *hold* (2)*, and for all* $k = 1,2,\dots$ *the system* (11) *satisfy all the conditions of Lemma* 3*. Then there exist* $\mu_3 \in (0,\mu_0)$, $\varepsilon_3(\mu) \in (0,\varepsilon_0)$ *such that* $\forall\, \mu \in (0,\mu_3)$, $\varepsilon \in (0,\varepsilon_3(\mu))$ *there exists the transformation of kind* (3)*, where* $Q(t,\varepsilon,\theta,\mu) \in F_2(m-1;\varepsilon_3(\mu);\theta)$*, which leads the system* (1) *to kind* (4)*, where the elements of diagonal matrix* $D(t,\varepsilon,\theta,\mu) \in F_2(m-1;\varepsilon_3(\mu);\theta)$ *are defined by formulas* (10)*.*