Kneser Solutions to Second Order Nonlinear Equations with Indefinite Weight

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1 Introduction

Consider the nonlinear differential equation

$$(a(t)\Phi(x'))' + b(t)F(x) = 0, \quad t \in [1,\infty), \tag{1.1}$$

where

$$\Phi(u) := |u|^{\alpha} \operatorname{sgn} u, \quad \alpha > 0.$$

We study the problem of the existence of *Kneser solutions*, that is solutions x such that

$$x(t) > 0, \quad x'(t) < 0 \ \text{ for } \ t \in [1,\infty), \tag{1.2}$$

satisfying the boundary conditions

$$x(1) = c > 0, \quad \lim_{t\to\infty} x(t) = 0. \tag{1.3}$$

We assume that the functions a, b are continuous functions on $[1,\infty)$, $a(t) > 0$, and

$$J_a = \int\limits_1^\infty \Psi\Big(\frac{1}{a(t)}\Big)\,dt < \infty,$$

where Ψ is the inverse function of Φ, that is $\Psi(u) := |u|^{1/\alpha} \operatorname{sgn} u$. The weight function b is bounded from above and is allowed to change sign (in)finite many times. The nonlinearity F is a continuous function on $[0,\infty)$ such that $F(u) > 0$ for $u > 0$ and

$$\limsup_{u\to 0+} \frac{F(u)}{\Phi(u)} < \infty. \tag{1.4}$$

This problem is motivated by [3] where some asymptotic BVPs are studied for (1.1) in case $F(u) = |u|^{\beta} \operatorname{sgn} u$, $\beta > 0$ and $b(t) \le 0$ for $t \ge 1$. There are few contributions to the solvability of the boundary value problems when the function b is allowed to change its sign. For example, the boundary value problem on the compact interval with the indefinite weight has been considered in [1].

In [4], our method used here is based on a fixed point theorem for operators defined in a Fréchet space stated in [2]. This approach does not require the explicit form of the fixed point operator but only good *a-priori* bounds. These bounds are obtained using the principal solutions of an associated linear or half-linear differential equations.

Our proofs are based on the following fixed point theorem.

Theorem 1 ([2]). *Consider the BVP on $[1,\infty)$,*

$$(a(t)\Phi(x'))' + b(t)F(x) = 0, \quad x \in S, \tag{1.5}$$

where S is a nonempty subset of the Fréchet space $C[1,\infty)$ of the continuous functions defined in $[1,\infty)$ endowed with the topology of uniform convergence on compact subsets of $[1,\infty)$.

Let G be a continuous function on $[0,\infty)\times[0,\infty)$ such that $F(d) = G(d,d)$ for any $d \in [0,\infty)$. Assume that there exist a nonempty, closed, convex and bounded subset $\Omega \subseteq C[1,\infty)$ and a bounded closed subset $S_1 \subseteq S \cap \Omega$ such that for any $u \in \Omega$ the BVP on $[1,\infty)$

$$(a(t)\Phi(x'))' + b(t)G(u(t),x(t)) = 0, \quad x \in S_1$$

admits a unique solution. Then the BVP (1.5) has at least a solution.

In the sequel, we introduce the notion of principal solution and disconjugacy for the half-linear equation

$$(a(t)\Phi(y'))' + \beta(t)\Phi(y) = 0, \tag{1.6}$$

where β is a continuous function for $t \ge 1$. When (1.6) is nonoscillatory, the notion of principal solution of (1.6) has been introduced in [7] by following the Riccati approach, see, also [6, Sections 2.2, 4.2]. Among all eventually different from zero solutions of the associated Riccati equation

$$w' + \beta(t) + R(t,w) = 0, \tag{1.7}$$

where

$$R(t,w) = \alpha|w|\Psi\Big(\frac{|w|}{a(t)}\Big),$$

there exists one, say w_x, which is continuable to infinity and is minimal in the sense that any other solution w of (1.7), which is continuable to infinity, satisfies $w_x(t) < w(t)$ as $t \to \infty$. This concept extends to the half-linear case the well-known notion of principal solution that was introduced in 1936 by W. Leighton and M. Morse for the linear case.

We recall that (1.6) is said to be *disconjugate* on an interval $I \subset [T,\infty)$ if any nontrivial solution of (1.6) has at most one zero on I. Equation (1.6) is disconjugate on $[T,\infty)$ if and only if it has the principal solution without zeros on (T,∞).

An important role in our considerations is played by a comparison theorem for the principal solutions of Sturm majorant and minorant half-linear equations established in [5]. It is worth to note that if $\alpha = 1$, the half-linear equation reduces to linear one and its principal solution can be characterized by the condition

$$\int\limits_1^\infty \frac{1}{a(t)x^2(t)}\,dt = \infty. \tag{1.8}$$

However, the integral characterization of the principal solution of half-linear equations remains an open problem. Hence, in the half-linear case a different approach has been used.

2 Existence and uniqueness theorem: case $\alpha = 1$

Consider nonlinear equation with the Sturm–Liouville operator

$$(a(t)x')' + b(t)F(x) = 0. \tag{2.1}$$

In addition to assumptions stated in Introduction, we also assume here that F is differentiable on $[0,\infty)$ with bounded nonnegative derivative, that is

$$0 \le \frac{dF(u)}{du} \le K \ \text{ for } \ u \ge 0, \tag{2.2}$$

and satisfies

$$\lim_{u\to 0^+} \frac{F(u)}{u} = k_0, \quad \lim_{u\to\infty} \frac{F(u)}{u} = k_\infty, \tag{2.3}$$

where $0 \le k_0 \ne k_\infty$.

The following result has been stated in [4, Theorem 3], see also Remark 5.

Theorem 2. *Let $B > 0$ be such that*

$$b(t) \le B \ \text{ on } \ [1,\infty)$$

and assume that the linear differential equation

$$v'' + \frac{BK}{a(t)}\, v = 0 \tag{2.4}$$

is disconjugate on $[1,\infty)$. Then, for any $c > 0$, equation (2.1) has a unique solution x satisfying (1.2) and (1.3). Moreover, such solution x satisfies (1.8).

Example. Consider the equation

$$(t^2x')' + \frac{1}{4}\cos\Big(\frac{\pi t}{2}\Big) F(x) = 0 \ \ (t \ge 1), \tag{2.5}$$

where

$$F(u) = \frac{u}{1+\sqrt{u}}\,.$$

Then F satisfies (2.2), (2.3), $K = 1$ and $b(t) \le 1/4$ for $t \ge 1$. Hence equation (2.4) becomes the Euler equation

$$v'' + \frac{1}{4t^2}\, v = 0 \ \ (t \ge 1),$$

which has a principal solution $v = \sqrt{t}$ and thus it is disconjugate on $[1,\infty)$. By Theorem 2, for any $c > 0$, equation (2.5) has a unique Kneser solution satisfying (1.2), (1.3) and (1.8).

3 Existence theorem in the general case

Denote by b_+, b_-, respectively, the positive and the negative part of b, i.e., $b_+(t) = \max\{b(t),0\}$, $b_-(t) = -\min\{b(t),0\}$. Thus $b(t) = b_+(t) - b_-(t)$.

Denote by $\widetilde{F}$ the function

$$\widetilde{F}(v) = \frac{F(v)}{\Phi(v)} \ \text{ on } \ (0,\infty). \tag{3.1}$$

In view of (1.4), the function $\widetilde{F}$ is bounded in the neighbourhood of zero.

Using Theorem 1 and asymptotic properties of the half-linear equations, we obtain from [5, Theorem 1] the following result.

Theorem 3. *Let $c > 0$ be fixed and M_c be such that*

$$\widetilde{F}(v) \le M_c \;\; on \;\; [0, c].$$

Assume that the half-linear differential equation

$$(a_1(t)\Phi(y'))' + \beta_1(t)\Phi(y) = 0, \tag{3.2}$$

where

$$a_1(t) \le a(t), \quad \beta_1(t) \ge M_c b_+(t) \;\; on \;\; t \ge 1, \tag{3.3}$$

has a principal solution which is positive decreasing on $[1, \infty)$.

Then, the BVP $(1.1), (1.3)$ *has at least one solution x if any of the following conditions holds:*

(i$_1$)

$$\lim_{T\to\infty} \int\limits_1^T |b(t)|\,\Phi\bigg(\int\limits_t^\infty \Psi\Big(\frac{1}{a(s)}\Big)\,ds\bigg)\,dt < \infty; \tag{3.4}$$

(i$_2$) *There exists $\bar{t} \ge 1$ such that $b_+(t) = 0$ for any $t \ge \bar{t}$.*

Moreover, if (i$_1$) *holds, such solution x satisfies*

$$\lim_{t\to\infty} \frac{x(t)}{\int\limits_t^\infty \Psi(a^{-1}(s))\,ds} = \ell, \;\; 0 < \ell < \infty. \tag{3.5}$$

Remark. A typical nonlinearity satisfying (1.4) is $F(u) = u^\beta$. A prototype of an half-linear equation (3.2) is the Euler type equation

$$(t^{1+\alpha}\Phi(y'))' + \Big(\frac{1}{1+\alpha}\Big)^{1+\alpha}\Phi(y) = 0. \tag{3.6}$$

From [6, Theorem 4.2.4], the function

$$y_0(t) = \Big(\frac{1}{1+\alpha}\Big)^{1/\alpha} t^{-1/(1+\alpha)}$$

is the principal solution of (3.6). Moreover, y_0 is positive decreasing on the interval $[1, \infty)$ and so (3.6) is disconjugate on the same interval. Other examples can be found in [5].

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