# On the Choice of Additional Initial Condition for Some Three-Level Difference Schemes

**Givi Berikelashvili**

*A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, Tbilisi, Georgia;*
*Department of Mathematics, Georgian Technical University, Tbilisi, Georgia*
*E-mail:* bergi@rmi.ge; berikela@yahoo.com

**Bidzina Midodashvili**

*Faculty of Exact and Natural Sciences, I. Javakhishvili Tbilisi State University, Tbilisi, Georgia*
*E-mail:* bidmid@hotmail.com

**Abstract.** In this paper we study an initial boundary-value problem for the Regularized Long Wave (RLW) equation. A three-level conservative difference scheme is constructed and investigated. For each new level the obtained algebraic equations are linear with respect to the values of unknown function.

## 1 Introduction

We consider one-dimensional RLW equation

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+\lambda u\,\frac{\partial u}{\partial x}-\mu\,\frac{\partial^3 u}{\partial x^2\partial t}=0, \tag{1.1}$$

with the physical boundary conditions $u\to 0$ as $x\to\pm\infty$. Here $u(x,t)$ represents the wave's amplitude, and $\lambda$ and $\mu$ are positive parameters.

This equation describes phenomena with weak nonlinearity and dispersion waves, including, for example, ion-acoustic and magnetohidrodynamic waves in plasma.

The main difficulties of numerical solution of (1.1) consist in physical domain boundless and nonlinearity of the equation, therefore, it is expedient to restrict the computational domain to a finite one. Suppose that the initial data $u_0(x)$ is compactly supported in a finite domain $(a,b)\subset\mathbb{R}$ which contains the compact support of $u(x,t)$.

We consider RLW equation (1.1) with the homogeneous boundary conditions

$$u(a,t)=0,\quad u(b,t)=0,\ \ 0<t\le T,$$

and the initial condition

$$u(x,0)=u_0(x),\ \ a\le x\le b.$$

## 2 Construction of difference scheme

The domain $[a,b]\times[0,T]$ is divided into rectangle grids by

$$x_i=a+ih,\ \ t_j=j\tau,\ \ i=1,2,\dots,n,\ \ j=0,1,2,\dots,J,$$

where $h=(b-a)/n$ and $\tau=T/J$ denote the spatial and temporal mesh sizes, respectively. For discrete functions defined on the mesh we use notation $U_i^j=U(x_i,t_j)$, $U_i^j\sim u(x_i,t_j)$.

In some cases, for simplicity and not implying vagueness, we omit some indices of the discrete function. We introduce fictitious values $U_{-1}^j$, $U_{n+1}^j$ which correspond to the abscissaes $x_{-1}=a-h$, $x_{n+1}=b+h$ and are defined by the equalities:

$$U_{-1}^j=0,\ \ U_{n+1}^j=0,\ \ j=0,1,2,\dots .$$

Let

$$Z_h^0=\big\{v=(v_i)\ |\ \ v_{-1}=v_0=v_n=v_{n+1}=0\big\}.$$

Define

$$\begin{gathered}
(U_i^j)_x=\frac{U_{i+1}^j-U_i^j}{h}\,,\quad (U_i^j)_{\overline{x}}=\frac{U_i^j-U_{i-1}^j}{h}\,,\\
(U_i^j)_{\dot{x}}=\frac{1}{2h}\,(U_{i+1}^j-U_{i-1}^j),\quad (U_i^j)_{\ddot{x}}=\frac{1}{4h}\,(U_{i+2}^j-U_{i-2}^j),\\
\overline{U}_i^{\,0}=\frac{U_i^1+U_i^0}{2}\,,\quad \overline{U}_i^{\,j}=\frac{U_i^{j+1}+U_i^{j-1}}{2}\ \text{ for }\ j\ge 1,\\
(U_i^j)_t=\frac{U_i^{j+1}-U_i^j}{\tau}\,,\quad (U_i^j)_{\overset{\circ}{t}}=\frac{1}{2\tau}\,(U_i^{j+1}-U_i^{j-1}).
\end{gathered}$$

Define the following averaging operators

$$\begin{gathered}
\dot{\mathcal{P}}u=\frac{1}{h^2}\int\limits_{x-h}^{x+h}\big(h-|x-\xi|\big)u(\xi,t)\,d\xi,\quad \ddot{\mathcal{P}}u=\frac{1}{4h^2}\int\limits_{x-2h}^{x+2h}\big(2h-|x-\xi|\big)u(\xi,t)\,d\xi,\\
\overset{\circ}{\mathcal{S}}u=\frac{1}{2\tau}\int\limits_{t-\tau}^{t+\tau}u(x,\zeta)\,d\zeta,\quad \widehat{\mathcal{S}}u=\frac{1}{\tau}\int\limits_{t}^{t+\tau}u(x,\zeta)\,d\zeta.
\end{gathered}$$

Let us consider some equalities connected with these operators

$$\dot{\mathcal{P}}\,\frac{\partial^2u}{\partial x^2}=u_{\overline{x}x},\quad \ddot{\mathcal{P}}\,\frac{\partial^2u}{\partial x^2}=u_{\dot{x}\dot{x}},\quad \overset{\circ}{\mathcal{S}}\,\frac{\partial u}{\partial t}=u_{\overset{\circ}{t}}.$$

It is easy to verify that

$$\dot{\mathcal{P}}u=u+\frac{h^2}{12}\,\frac{\partial^2u}{\partial x^2}+O(h^4),\quad \ddot{\mathcal{P}}u=u+\frac{4h^2}{12}\,\frac{\partial^2u}{\partial x^2}+O(h^4),$$

whence

$$(4\dot{\mathcal{P}}-\ddot{\mathcal{P}})u=3u+O(h^4).$$

Let us act on (1.1) with the operator

$$\frac{1}{3}\,(4\dot{\mathcal{P}}-\ddot{\mathcal{P}})\overset{\circ}{\mathcal{S}}.$$

Notice that

$$\begin{gathered}
\frac{1}{3}\,(4\dot{\mathcal{P}}-\ddot{\mathcal{P}})\overset{\circ}{\mathcal{S}}\,\frac{\partial u}{\partial t}=\frac{1}{3}\,(4\dot{\mathcal{P}}-\ddot{\mathcal{P}})u_{\overset{\circ}{t}}=u_{\overset{\circ}{t}}+O(h^4),\\
\frac{1}{3}\,(4\dot{\mathcal{P}}-\ddot{\mathcal{P}})\overset{\circ}{\mathcal{S}}\,\frac{\partial^3u}{\partial x^2\partial t}=\frac{1}{3}\,(4\dot{\mathcal{P}}-\ddot{\mathcal{P}})\Big(\frac{\partial^2u}{\partial x^2}\Big)_{\overset{\circ}{t}}=\frac{1}{3}\,(4u_{\overline{x}x\overset{\circ}{t}}-u_{\dot{x}\dot{x}\overset{\circ}{t}}).
\end{gathered}$$

Further,

$$\frac{1}{3}\,(4\dot{\mathcal{P}}-\ddot{\mathcal{P}})\overset{\circ}{\mathcal{S}}\,\frac{\partial u}{\partial x}=\frac{1}{3}\,(4\dot{\mathcal{P}}-\ddot{\mathcal{P}})\,\frac{\partial \overline{u}}{\partial x}=\frac{1}{3}\,(4\overline{u}_{\dot{x}}-\overline{u}_{\ddot{x}})+O(\tau^2+h^4).$$

Finally, after some transformations we have

$$\begin{aligned}(4\dot{\mathcal{P}}-\ddot{\mathcal{P}})\overset{\circ}{\mathcal{S}}\Big(u\,\frac{\partial u}{\partial x}\Big)&=(4\dot{\mathcal{P}}-\ddot{\mathcal{P}})\Big(u\,\frac{\partial \overline{u}}{\partial x}\Big)+O(\tau^2)=3u\frac{\partial \overline{u}}{\partial x}+O(\tau^2+h^4)\\&=\frac{4}{3}\,\big[\overline{u}_{\dot{x}}u+(\overline{u}u)_{\dot{x}}\big]-\frac{1}{3}\,\big[\overline{u}_{\ddot{x}}u+(\overline{u}u)_{\ddot{x}}\big]+O(\tau^2+h^4).\end{aligned}$$

Thus, we have the difference scheme

$$\begin{aligned}(U_i^j)_{\overset{\circ}{t}}+\Big(\frac{4}{3}\,(\overline{U}_i^j)_{\dot{x}}-\frac{1}{3}\,(\overline{U}_i^j)_{\ddot{x}}\Big)+\frac{4\lambda}{9}\,\kappa_1(\overline{U}_i^j,U_i^j)-\frac{\lambda}{9}\,\kappa_2(\overline{U}_i^j,U_i^j)\\-\mu\Big(\frac{4}{3}\,(U_i^j)_{\overline{x}x\overset{\circ}{t}}-\frac{1}{3}\,(U_i^j)_{\dot{x}\dot{x}\overset{\circ}{t}}\Big)=0,\ \ i=1,2,\dots,n-1;\ \ j=1,2,\dots,J-1,\ \ U\in Z_h^0,\end{aligned}\tag{2.1}$$

where

$$\kappa_1(U,V)=U_{\dot{x}}V+(UV)_{\dot{x}},\quad \kappa_2(U,V)=U_{\ddot{x}}V+(UV)_{\ddot{x}}.$$

The additional initial conditions (the values of unknown function on the first level) is found with two-level linear scheme:

$$\begin{aligned}(U_i^0)_t+\Big(\frac{4}{3}\,(\overline{U}_i^0)_{\dot{x}}-\frac{1}{3}\,(\overline{U}_i^0)_{\ddot{x}}\Big)+\frac{4\lambda}{9}\,\kappa_1(\overline{U}_i^0,U_i^0)\\-\frac{\lambda}{9}\,\kappa_2(\overline{U}_i^0,U_i^0)-\mu\Big(\frac{4}{3}\,(U_i^0)_{\overline{x}xt}-\frac{1}{3}\,(U_i^0)_{\dot{x}\dot{x}t}\Big)=0,\ \ i=1,2,\dots,n-1.\end{aligned}\tag{2.2}$$

It is proved that the difference scheme (2.1), (2.2) is uniquely solvable, conservative, absolutely stable and converges with rate $O(\tau^2+h^4)$.

Equations (2.2) are especially notable. Some authors suggest that this is the approximation of the differential equation using initial conditions and attempt to receive an approximation with the same order truncation error as for the differential equation. We think that (2.2) is an approximation of the initial conditions for the first level using the differential equation. It must be required an appropriate order of approximation of initial data. This is confirmed in our papers (see, e.g. [1–3]).

## References

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