Analogue of the Erugin Theorem on the Absence of Strongly Irregular Periodic Solutions of Two-dimensional Linear Discrete Periodic System

M. S. Belokursky

Department of Mathematical Analysis and Differential Equations,
F. Scorina Gomel State University, Gomel, Belarus
E-mail: drakonsm@ya.ru

A. K. Demenchuk

Department of Differential Equations, Institute of Mathematics,
National Academy of Science of Belarus, Minsk, Belarus
E-mail: ddemenchuk@im.bas-net.by

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ the sets of natural, integer and real numbers, respectively, $z = (z_n) = (z(n))$ $(n \in \mathbb{N}) - l$-dimensional vector function (sequence), defined on $\mathbb{N}$ with values in $\mathbb{R}^l$, i.e. $z : \mathbb{N} \to \mathbb{R}^l$. The set of such sequences is denoted by S^l. Following [1, p. 69] we introduce the definition.

Definition 1. A sequence $z \in S^l$ is called periodic with a period $\omega \in \mathbb{N}$ (ω-periodic) if for any $n \in \mathbb{N}$ the equality $z_{n+\omega} = z_n$ holds.

Naturally, if the number ω is the period of the sequence z, then its multiples will also be the periods of this sequence, i.e. for any $n \in \mathbb{N}$, $m \in \mathbb{N}$, we have $z(n + m\omega) = z(n)$. Therefore, in the future, under the period of the sequence, as a rule, we will understand the smallest of the periods. In this case, in particular, any constant scalar sequence will be 1-periodic. The set of l-dimensional ω-periodic sequences is denoted by PS^l_ω.

Periodic sequences under certain conditions can be solutions of discrete (difference) systems. The problem of the existence and construction of periodic solutions of discrete equations and systems is considered in a sufficiently large number of papers [1, 4, 6] etc. In these papers solutions are mainly studied, the period of which coincides with the period of the equation. The results obtained in this direction are in many respects similar to the corresponding results for ordinary differential equations. However, in some cases there are significant differences. Note one of them.

As it is known [8], a nonlinear scalar periodic ordinary differential equation does not have nonconstant periodic solutions such that the periods of the solution and equation are incommensurable. Moreover, N. P. Erugin proved in [5] that such solutions are absent in the linear nonstationary periodic system of two equations. It is interesting to investigate such questions for discrete equations and systems. For this purpose, we consider the system

$$x_{n+1} = X(x_n, y_n, n), \quad y_{n+1} = Y(x_n, y_n, n), \;\; n \in \mathbb{N}, \;\; \operatorname{col}(x, y) \in S^2, \tag{1}$$

the right side of which is ω-periodic, i.e. there exists the smallest $\omega \in \mathbb{N}$ such that for any fixed $n_0 \in \mathbb{N}$ equalities $X(x_{n_0}, y_{n_0}, n + \omega) = X(x_{n_0}, y_{n_0}, n)$, $Y(x_{n_0}, y_{n_0}, n + \omega) = Y(x_{n_0}, y_{n_0}, n)$ hold for all $n \in \mathbb{N}$. Further, the period of the system of the form (1) is understood as the period of its right side.

Analogous to [2], we introduce the following

Definition 2. A periodic solution with a period of the system (1) such that the numbers ω and Ω are coprime, we will call strongly irregular.

We note that the paper [7] shows the following: under certain conditions, the scalar discrete equation can admit a strongly irregular periodic solution. Indeed, let σ be an arbitrary odd number and $(h_n) \in PS^1_\sigma$. Take the discrete equation

$$x_{n+1} = -x_n - (1 - x_n^2)h_n. \tag{2}$$

The equation (2) has a solution

$$x_n = (-1)^n \tag{3}$$

with period $\Omega = 2$. As the numbers σ and Ω coprime, by Definition 2, the periodic solution (3) of the equation (2) is strongly irregular.

Thus, Massera's theorem [8] on the absence of strongly irregular periodic solutions for a scalar ordinary equation for difference equations, generally speaking, has no complete analog for discrete equations. An analogue of Massera's theorem for linear difference equations was obtained in [3]. In particular, it is shown that the scalar linear homogeneous periodic nonstationary discrete equation of the first order has not strongly irregular periodic solutions different from the constants.

It is quite natural to raise the question for the two-dimensional case: is there an analogue of the above theorem by N. P. Erugin on the two-dimensional linear system (1)

$$x_{n+1} = a_n x_n + b_n y_n, \quad y_{n+1} = c_n x_n + d_n y_n, \ \ n \in \mathbb{N}, \ \ x \in S^1, \ \ y \in S^1, \tag{4}$$

where the coefficient matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is ω-periodic, i.e. $A(n+\omega) = A(n)$ for all $n \in \mathbb{N}$ and at least one of its elements is different from the constant? As the following example shows, the answer to this question is generally negative. Indeed, take a linear discrete system

$$x_{n+1} = -x_n + b_n y_n, \quad y_{n+1} = d_n y_n, \ \ n \in \mathbb{N}, \ \ (b_n) \in PS^1_\omega, \ \ (d_n) \in PS^1_\omega, \tag{5}$$

where at least one of the coefficients (b_n), (d_n) is different from the constant, i.e. $\omega \geq 2$, and the greatest common divisor of numbers 2 and ω is 1. The system (5) has a periodic solution

$$x_n = (-1)^n, \quad y_n = 0, \ \ n \in \mathbb{N}. \tag{6}$$

The period of the solution (6) is coprime with the period of the system (5).

Our goal is to distinguish a class of linear two-dimensional discrete systems that have not strongly irregular periodic solutions.

Further, we say that the columns $H^{(1)}(n), \dots, H^{(k)}(n)$ of some matrix $H(n)$, $n \in \mathbb{N}$ are linearly independent if the identity

$$\alpha_1 H^{(1)}(n) + \dots + \alpha_k H^{(k)}(n) \equiv 0, \ \ n \in \mathbb{N}, \ \ \alpha_1, \dots, \alpha_k \in \mathbb{R}$$

holds if and only if $\alpha_1 = \dots = \alpha_k = 0$. Through $\operatorname{rank}_{\mathrm{col}} H$ denote the column rank of the matrix $H(n)$, $n \in \mathbb{N}$, i.e. the largest number of its linearly independent columns.

Suppose that the system (4) has a strongly irregular Ω-periodic solution

$$x_n = \varphi_n, \quad y_n = \psi_n, \quad \varphi(n+\Omega) = \varphi(n), \quad \psi(n+\Omega) = \psi(n), \ \ n \in \mathbb{N}, \tag{7}$$

where ω and Ω are coprime and $\Omega \geq 2$. This means that

$$\varphi_{n+1} \equiv a_n\varphi_n + b_n\psi_n, \quad \psi_{n+1} \equiv c_n\varphi_n + d_n\psi_n, \ \ n \in \mathbb{N}. \tag{8}$$

As the identities (8) are true for all $n \in \mathbb{N}$, there are also true

$$\varphi_{n+1+\Omega} \equiv a_{n+\Omega}\varphi_{n+\Omega} + b_{n+\Omega}\psi_{n+\Omega}, \quad \psi_{n+1} \equiv c_{n+\Omega}\varphi_{n+\Omega} + d_{n+\Omega}\psi_{n+\Omega}, \ \ n \in \mathbb{N}. \tag{9}$$

By virtue of the Ω-periodicity of functions φ_n, ψ_n, the identities (9) take the following form

$$\varphi_{n+1} \equiv a_{n+\Omega}\varphi_n + b_{n+\Omega}\psi_n, \quad \psi_{n+1} \equiv c_{n+\Omega}\varphi_n + d_{n+\Omega}\psi_n, \ \ n \in \mathbb{N}. \tag{10}$$

The identities (8), (10) implies the following

$$\begin{aligned} &(a_{n+\Omega} - a_n)\varphi_n + (b_{n+\Omega} - b_n)\psi \equiv p^{(11)}(n)\varphi_n + p^{(12)}(n)\psi_n \equiv 0, \\ &(c_{n+\Omega} - c_n)\varphi_n + (d_{n+\Omega} - d_n)\psi \equiv p^{(21)}(n)\varphi_n + p^{(22)}(n)\psi_n \equiv 0, \end{aligned} \quad n \in \mathbb{N}. \tag{11}$$

We form a matrix

$$P(n) = \begin{bmatrix} p^{(11)}(n) & p^{(12)}(n) \\ p^{(21)}(n) & p^{(22)}(n) \end{bmatrix}, \ \ n \in \mathbb{N}.$$

We denote by $P^{(j)}(n)$, $n \in \mathbb{N}$, $j = 1, 2$ the columns of this matrix. As $P(n) = A(n+\Omega) - A(n)$ and $A(n+\omega) \equiv A(n)$, $n \in \mathbb{N}$, the matrix function P is ω-periodic.

We show that the columns $P^{(1)}(n)$ and $P^{(2)}(n)$ are linearly dependent, i.e. there are exist such $\alpha_0, \beta_0 \in \mathbb{R}$, $\alpha_0^2 + \beta_0^2 \neq 0$, that $\alpha_0 P^{(1)}(n) + \beta_0 P^{(2)}(n) \equiv 0$, $n \in \mathbb{N}$. According to the assumption, at least one of the functions $x = \varphi$, $y = \psi$ is nonstationary. Therefore, there exists $n_0 \in \mathbb{N}$ for which the inequality $\varphi_{n_0}^2 + \psi_{n_0}^2 \neq 0$ holds. The identities (11) imply the justice of equalities

$$\varphi_{n_0+m\Omega}P^{(1)}(n_0 + m\Omega) + \psi_{n_0+m\Omega}P^{(2)}(n_0 + m\Omega) = 0, \ \ m \in \mathbb{N},$$

from which, on the basis of the Ω-periodicity of functions φ, ψ, we obtain the equality

$$\varphi_{n_0}P^{(1)}(n_0 + m\Omega) + \psi_{n_0}P^{(2)}(n_0 + m\Omega) = 0, \ \ m \in \mathbb{N}. \tag{12}$$

As the matrix P has a period ω, the equality (12) can be written as

$$\varphi_{n_0}P^{(1)}(n_0 + m\Omega + k\omega) + \psi_{n_0}P^{(2)}(n_0 + m\Omega + k\omega) = 0, \ \ k, m \in \mathbb{N}. \tag{13}$$

Since k, m are an arbitrary natural numbers and least common multiple of ω and Ω is 1, for any $n \in \mathbb{N}$ there exist such k, m that the equation $n = n_0 + m\Omega + k\omega$ holds. Therefore, $P^{(j)}(n_0 + m\Omega + k\omega) = P^{(j)}(n)$, $n \in \mathbb{N}$, $j = 1, 2$ for $k, m \in \mathbb{N}$. Hence, from the equations (13) we obtain

$$\varphi_{n_0}P^{(1)}(n) + \psi_{n_0}P^{(2)}(n) = 0, \ \ n \in \mathbb{N}. \tag{14}$$

By virtue of the fact that $\varphi_{n_0}^2 + \psi_{n_0}^2 \neq 0$, the identity (14) means that the columns of the matrix $P(n)$, $n \in \mathbb{N}$ are linearly dependent.

So, we have proved the following

Theorem. *If the system* (4) *has a nonstationary periodic solution such that the solution period is coprime with the system's period, then the columns of the matrix are linearly dependent.*

Corollary. *If the matrix $P(n)$, $n \in \mathbb{N}$ has a complete column rank, i.e.* $\mathrm{rank}_{\mathrm{col}} P = 2$*, the system* (4) *has not nonstationary strongly irregular periodic solutions.*

Remark 1. As shown above, the discrete periodic system (5) has a strongly irregular 2-periodic solution (6). The matrix $P(n)$, $n \in \mathbb{N}$ for this system has the form

$$P(n) = \begin{bmatrix} 0 & b(n+2) - b(n) \\ 0 & d(n+2) - d(n) \end{bmatrix}, \quad n \in \mathbb{N}. \tag{15}$$

The columns of this matrix are linearly dependent and its column rank in generall case is one.

Remark 2. In general, the linear dependence of the columns and rows of a discrete matrix is not equivalent. This is particularly confirmed by the example (15), where the matrix rows can be linearly dependent only if

$$b(n+2) - b(n) \equiv l(d(n+2) - d(n)), \quad l \in \mathbb{R}.$$

References

[1] R. P. Agarwal, *Difference Equations and Inequalities. Theory, Methods, and Applications.* Monographs and Textbooks in Pure and Applied Mathematics, 155. Marcel Dekker, Inc., New York, 1992.

[2] A. K. Demenchuk, *Asynchronous Oscillations in Differential Systems. Conditions for the Existence and Control.* (Russian) Lambert Academic Publishing, Saarbrücken, 2012.

[3] A. K. Demenchuk, On the strongly irregular periodic solutions of a linear homogenous discrete equations of the first order. (Russian) *Doklady NAN Belarusi* **62** (2018), no. 3, 263–267.

[4] S. Elaydi, *An Introduction to Difference Equations.* Third edition. Undergraduate Texts in Mathematics. Springer, New York, 2005.

[5] N. P. Erugin, On the periodic solutions of a linear homogeneous system of differential equations. (Russian) *Dokl. Akad. Nauk BSSR* **6** (1962), 407–410.

[6] K. Janglajew and E. Schmeidel, Periodicity of solutions of nonhomogeneous linear difference equations. *Adv. Difference Equ.* **2012**, 2012:195, 11 pp.

[7] A. V. Lasunsky, On the period of solutions of a discrete periodic logistic equation. (Russian) *Trudy Karelskogo Nauchnogo Tsentra RAN*, 2012, no. 5, 44–48.

[8] J. L. Massera, Remarks on the periodic solutions of differential equations. (Spanish) [Observaciones sobre les soluciones periodicas de ecuaciones diferenciales]. *Bol. Fac. Ingen. Montevideo* **4** (1950) (Año 14), 37–45 = 43–53 Facultad de Ingeniería Montevideo. Publ. Inst. Mat. Estadística **2**, 43–53 (1950).