On Dimensions of Subspaces Defined by Lyapunov Exponents of Families of Linear Differential Systems

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1 Introduction

Let M be a metric space. For a given positive integer n consider a family of linear differential systems depending on the parameter $\mu \in M$:

$$\dot{x} = A(t,\mu)x, \quad x \in \mathbb{R}^n, \;\; t \in \mathbb{R}_+ \equiv [0,+\infty), \tag{1.1}$$

such that the matrix function $A(\,\cdot\,,\mu)\colon \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is continuous and bounded for each fixed $\mu \in M$ (generally speaking, the bound being dependent on μ). Therefore, fixing a value of the parameter $\mu \in M$ in the family (1.1), we obtain a linear differential system with continuous coefficients bounded on the semiaxis. The Lyapunov exponents of this system are denoted by $\lambda_1(\mu;A) \leqslant \cdots \leqslant \lambda_n(\mu;A)$. Thus for each $k = \overline{1,n}$ we get the function $\lambda_k(\,\cdot\,;A)\colon M \to \mathbb{R}$, which is called the k-th Lyapunov exponent of the family (1.1), and the vector function $\Lambda(\,\cdot\,;A)\colon M \to \mathbb{R}^n$ defined by $\Lambda(\mu;A) = (\lambda_1(\mu;A),\dots,\lambda_n(\mu;A))^\top$.

In the theory of Lyapunov exponents, a family of matrix functions $A(\,\cdot\,,\mu)$, $\mu \in M$ (as stated, all functions are continuous and bounded on the semiaxis), is considered under one of the following two natural assumptions: that the family is continuous either **a)** in the compact-open topology, or **b)** in the uniform topology. The condition **a)** is equivalent to the fact that if a sequence $(\mu_k)_{k\in\mathbb{N}}$ of points from M converges to a point μ_0, then the sequence of functions $A(t,\mu_k)$ of the variable $t \geqslant 0$ converges to the function $A(t,\mu_0)$ as $k \to +\infty$ uniformly on each segment $[0,T] \subset \mathbb{R}_+$, while the condition **b)** is equivalent to the fact that this convergence is uniform over the whole semiaxis $\mathbb{R}_+$. Denote the class of families (1.1) that are continuous in the compact-open topology by $\mathcal{C}^n(M)$ and the class of those that are continuous in the uniform topology by $\mathcal{U}^n(M)$. It is clear that $\mathcal{U}^n(M) \subset \mathcal{C}^n(M)$. In what follows, we shall identify families (1.1) with the matrix-functions $A(\,\cdot\,,\,\cdot\,)$ defining them, and therefore write $A \in \mathcal{C}^n(M)$ or $A \in \mathcal{U}^n(M)$.

For families (1.1) V. M. Millionshchikov stated [9] the problem of description of their Lyapunov exponents as functions of a parameter. In other words, this problem is formulated as follows: for each $n \in \mathbb{N}$, $k = \overline{1,n}$, and metric space M describe the following classes of functions:

$$\Lambda_k(M;n,\mathcal{C}) = \big\{\lambda_k(\,\cdot\,;A):\; A \in \mathcal{C}^n(M)\big\} \;\text{ and }\; \Lambda_k(M;n,\mathcal{U}) = \big\{\lambda_k(\,\cdot\,;A):\; A \in \mathcal{U}^n(M)\big\}. \tag{1.2}$$

V. M. Millionshchikov proved that for any metric space M and family $A \in \mathcal{C}^n(M)$ each of the Lyapunov exponents $\lambda_k(\,\cdot\,;A)$ can be represented as the limit of a decreasing sequence of functions of the first Baire class. In particular, this implies that $\lambda_k(\,\cdot\,;A)$ is a function of the second Baire class on this space (this assertion followed from the essentially more general Millionshchikov theorem obtained by him in [8]). M. I. Rakhimberdiev proved [10] that the number of Baire class in the description above cannot be reduced even in the case of Lyapunov exponents of families from $\mathcal{U}^n(M)$. However, the problem of a complete description of the classes (1.2) until recently remained unsolved, the solution have been obtained in [6] and [4].

The description of the classes (1.2) is a special case of a more general problem – to describe for each $n \in \mathbb{N}$ and metric space M the following classes of vector functions:

$$\Lambda(M;n,\mathcal{C}) = \big\{\Lambda(\,\cdot\,;A):\ A \in \mathcal{C}^n(M)\big\} \ \text{ and } \ \Lambda(M;n,\mathcal{U}) = \big\{\Lambda(\,\cdot\,;A):\ A \in \mathcal{U}^n(M)\big\}. \tag{1.3}$$

For further discourse note that in the case $n = 1$, the description of the second of the classes (1.3) (i.e., of the class $\Lambda(M;1,\mathcal{U}) = \Lambda_1(M;1,\mathcal{U})$) is obvious: it consists of all continuous functions $M \to \mathbb{R}$.

Before presenting the main results on the description of the classes (1.2) and (1.3), recall the necessary definitions of the descriptive set theory [5, p. 267]. Let $\mathfrak{M}$ and $\mathfrak{N}$ be sets consisting of subsets of the space M. A function $f\colon M \to \mathbb{R}$ belongs to the class $(\mathfrak{M}, {}^*)$ if for any $r \in \mathbb{R}$ the preimage $f^{-1}((r,+\infty))$ of the interval $(r,+\infty)$ belongs to $\mathfrak{M}$. A function $f\colon M \to \mathbb{R}$ belongs to the class $({}^*, \mathfrak{N})$ if for any $r \in \mathbb{R}$ the preimage $f^{-1}([r,+\infty))$ of the half-interval $[r,+\infty)$ belongs to $\mathfrak{N}$. Finally, a function f belongs to the class $(\mathfrak{M}, \mathfrak{N})$ if it belongs to both classes $(\mathfrak{M}, {}^*)$ and $({}^*, \mathfrak{N})$.

For any $n \in \mathbb{N}$, $k = \overline{1,n}$, and metric space M, the classes $\Lambda_k(M;n,\mathcal{C})$ are described in [6] – a function $f\colon M \to \mathbb{R}$ belongs to the class $\Lambda_k(M;n,\mathcal{C})$ if and only if it: 1) belongs to the class $({}^*, G_\delta)$ and 2) has an upper semi-continuous minorant. For any $n \geqslant 2$, $k = \overline{1,n}$, and metric space M, the description of the classes $\Lambda_k(M;n,\mathcal{U})$ is obtained in [4]: a function $f\colon M \to \mathbb{R}$ belongs to the class $\Lambda_k(M;n,\mathcal{U})$ if and only if it satisfies the condition 1) and the condition 2′) it has continuous minorant and majorant. As can be seen from the formulations above, the descriptions of the classes $\Lambda_k(M;n,\mathcal{C})$ and $\Lambda_k(M;n,\mathcal{U})$ are similar, however, their proofs differ quite significantly. For any $n \in \mathbb{N}$, $k = \overline{1,n}$, and metric space M, the class $\Lambda(M;n,\mathcal{C})$ is described in [6], and the description of the class $\Lambda(M;n,\mathcal{U})$ was announced in [1] (the full proof is given in [2]). Moreover, the description of both classes (1.3) is obtained by adding to the conditions 1) and 2) (respectively, to 1) and 2′)), which are necessary since $\mathcal{U}^n(M) \subset \mathcal{C}^n(M)$, the inequalities $f_1(\mu) \leqslant \dots \leqslant f_n(\mu)$ for all $\mu \in M$. The latter inequalities obviously follow from the definition of the vector function $\Lambda(\,\cdot\,;A)$.

Let us emphasize that the description of the class $\Lambda(M;n,\mathcal{U})$ required for its proof an approach different from those used in [4, 6]. As noted above, the key part in the description of the class $\Lambda(M;n,\mathcal{U})$ is a (constructive) proof of the sufficiency of the conditions. Let us formulate this description [1, 2], since the results given below are closely related to it.

Theorem. *Let M be a metric space, an integer $n \geqslant 2$, and all components of a vector function $(f_1,\dots,f_n)^\top\colon M \to \mathbb{R}^n$ belong to the class $({}^*, G_\delta)$, have continuous minorant and majorant and satisfy the inequalities $f_1(\mu) \leqslant \dots \leqslant f_n(\mu)$ for all $\mu \in M$. Then there exists a family $A \in \mathcal{U}^n(M)$ such that $\Lambda(\,\cdot\,;A) = (f_1,\dots,f_n)^\top$.*

If the given vector function is bounded:

$$\sup\Big\{\big\|(f_1(\mu),\dots,f_n(\mu))^\top\big\|:\ \mu \in M\Big\} < +\infty,$$

then the statement of the above theorem can be significantly strengthened. Denote by $\mathcal{Q}^n(M)$ the class of families (1.1) of the form $A(t,\mu) = B(t) + Q(t,\mu)$, $t \in \mathbb{R}_+$, $\mu \in M$, where $B(t)$ is a bounded $n \times n$ matrix, and $Q(t,\mu)$ is a bounded $n \times n$ matrix vanishing as $t \to +\infty$ uniformly with respect to μ.

The proof of the preceding theorem implies the following

Corollary 1. *For any metric space M, integer $n \geqslant 2$, and vector function $(f_1, \dots, f_n)^\top : M \to \mathbb{R}^n$ whose components belong to the class $({}^*, G_\delta)$, are bounded and satisfy the inequalities $f_1(\mu) \leqslant \dots \leqslant f_n(\mu)$ for all $\mu \in M$, there exists a family $A \in \mathcal{Q}^n(M)$ such that $\Lambda(\,\cdot\,; A) = (f_1, \dots, f_n)^\top$.*

Let us give some more corollaries of the theorem presented here, which answer a number of open questions.

V. M. Millionshchikov proved [8] that if M is a complete metric space, then for a family $A \in \mathcal{C}^n(M)$ the set $US_i(A)$ of upper semicontinuity points of the function $\lambda_i(\,\cdot\,; A)$ contains a dense G_δ-set for each $i = \overline{1, n}$. In other words, the upper semicontinuity of these functions is Baire typical in the space M. This statement is not true for the lower semicontinuity: in [11] for each $n \geqslant 1$ there is constructed a family $A \in \mathcal{C}^n([0, 1])$ such that the set $LS_i(A)$ of lower semicontinuity points of the function $\lambda_i(\,\cdot\,; A)$, $i = \overline{1, n}$, is empty. A complete description of the n-tuples $(LS_1(A), \dots, LS_n(A))$ for any metric space M and a complete description of the n-tuples $(US_1(A), \dots, US_n(A))$ for any complete metric space M are obtained in [7] for the families $A \in \mathcal{C}^n(M)$. A family $A \in \mathcal{U}^n([0, 1])$ for which the set $LS_i(A)$ is empty is constructed in [13] for any $n \geqslant 2$ and $i = \overline{1, n}$. Later, using the ideas of that paper and the results of [7], a complete description of the sets $LS_i(A)$, $i = \overline{1, n}$, for any metric space M and a complete description of the sets $US_i(A)$, $i = \overline{1, n}$, for any complete metric space M were obtained in [3] for the families $A \in \mathcal{U}^n(M)$.

Using the main theorem we can give a complete description of the n-tuples $(LS_1(A), \dots, LS_n(A))$ for any metric space M and a complete description of the n-tuples $(US_1(A), \dots, US_n(A))$ for any complete metric space M for the families $A \in \mathcal{U}^n(M)$ thus giving an answer to the problem stated in [3].

Corollary 2. *For any integer $n \geqslant 2$ and metric space M, an n-tuple $(M_1, \dots, M_n)$ of subsets of M is the n-tuple of the lower semicontinuity sets of the Lyapunov exponents of some family $A \in \mathcal{U}^n(M)$ (i.e., $M_i = LS_i(A)$, $i = \overline{1, n}$) if and only if each set M_i, $i = \overline{1, n}$, is $F_{\sigma\delta}$ and contains all isolated points of M. Moreover, in cases where such a family exists, it can be chosen from the class $\mathcal{Q}^n(M)$.*

Corollary 3. *For any integer $n \geqslant 2$ and complete metric space M, an n-tuple $(M_1, \dots, M_n)$ of subsets of M is the n-tuple of the upper semicontinuity sets of the Lyapunov exponents of some family $A \in \mathcal{U}^n(M)$ (i.e., $M_i = US_i(A)$, $i = \overline{1, n}$) if and only if each set M_i, $i = \overline{1, n}$, is a dense G_δ-set in M. Moreover, in cases where such a family exists, it can be chosen from the class $\mathcal{Q}^n(M)$.*

For each $\mu \in M$ denote by $S(\mu; A)$ the vector space of solutions of the system (1.1). As is well known, the sets $L_\alpha(\mu; A) \stackrel{\text{def}}{=} \{x \in S(\mu; A) :\ \lambda[x] < \alpha\}$ and $N_\alpha(\mu; A) \stackrel{\text{def}}{=} \{x \in S(\mu; A) :\ \lambda[x] \le \alpha\}$ are vector subspaces of the space $S(\mu; A)$ for any $\alpha \in \mathbb{R}$. Denote their dimensions by $d_\alpha(\mu; A)$ and $D_\alpha(\mu; A)$ respectively. Next we consider the natural question: what are the functions $\mu \mapsto d_\alpha(\mu; A)$ and $\mu \mapsto D_\alpha(\mu; A)$? A. N. Vetokhin proved [12] that if M is the space of all linear n-dimensional systems endowed with either of the topologies: compact-open or uniform, and the family (1.1) is defined by the equality $A(t, \mu) = \mu(t)$, $\mu \in M$, $t \in \mathbb{R}_+$, then the first function belongs exactly to the second Baire class, and the second one belongs exactly to the third Baire class.

The following statements contain a complete description of the classes $\{d_\alpha(\mu; A) :\ A \in \mathcal{C}^n(M)\}$, $\{d_\alpha(\mu; A) :\ A \in \mathcal{U}^n(M)\}$, $\{D_\alpha(\mu; A) :\ A \in \mathcal{C}^n(M)\}$, and $\{D_\alpha(\mu; A) :\ A \in \mathcal{U}^n(M)\}$ for any metric space M and numbers $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$.

Corollary 4. *Let an arbitrary metric space M and numbers $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, and a function $f\colon M \to \{0, \dots, n\}$ be given. Then the equality $f = d_\alpha(\,\cdot\,; A)$ $(f = D_\alpha(\,\cdot\,; A))$ holds for some family $A \in \mathcal{C}^n(M)$ if and only if f belongs to the class (F_σ, F_σ) (respectively, to the class $(F_{\sigma\delta}, F_{\sigma\delta})$).*

Corollary 5. *Let an arbitrary metric space M and numbers $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, and a function $f\colon M \to \{0,\dots,n\}$ be given. Then the equality $f = d_\alpha(\,\cdot\,;A)$ $(f = D_\alpha(\,\cdot\,;A))$ holds for some family $A \in \mathcal{U}^n(M)$ if and only if*

1) *in the case $n \geqslant 2$, the function f belongs to the class (F_σ, F_σ) (respectively, $(F_{\sigma\delta}, F_{\sigma\delta})$);*

2) *in the case $n = 1$, the function f is lower semicontinuous (respectively, upper semicontinuous).*

Moreover, for $n \geqslant 2$, if such a family exists, then it can be chosen from the class $\mathcal{Q}^n(M)$.

Corollaries 4 and 5 allow us to describe the sets of semicontinuity of functions $d_\alpha(\,\cdot\,;A)$ and $D_\alpha(\,\cdot\,;A)$ for families $A \in \mathcal{C}^n(M)$ and $A \in \mathcal{U}^n(M)$.

Corollary 6. *Let an arbitrary metric space M and numbers $\alpha \in \mathbb{R}$, and $n \geqslant 2$ $(n \geqslant 1)$ be given. Then a set $S \subset M$ is the set of lower semicontinuity points of the function $d_\alpha(\,\cdot\,;A)$ for some family $A \in \mathcal{U}^n(M)$ $(A \in \mathcal{C}^n(M))$ if and only if S is a dense G_δ-subset. A set $S \subset M$ is the set of upper semicontinuity points of the function $d_\alpha(\,\cdot\,;A)$ for some family $A \in \mathcal{U}^n(M)$ $(A \in \mathcal{C}^n(M))$ if and only if S is a dense F_σ-subset. Moreover, for $n \geqslant 2$, if the mentioned family exists, then it can be chosen from the class $\mathcal{Q}^n(M)$.*

Corollary 7. *Let an arbitrary metric space M and numbers $\alpha \in \mathbb{R}$, and $n \geqslant 2$ $(n \geqslant 1)$ be given. Then a set $S \subset M$ is the set of lower semicontinuity points of the function $D_\alpha(\,\cdot\,;A)$ for some family $A \in \mathcal{U}^n(M)$ $(A \in \mathcal{C}^n(M))$ if and only if S is a dense $F_{\sigma\delta}$-subset. A set $S \subset M$ is the set of upper semicontinuity points of the function $D_\alpha(\,\cdot\,;A)$ for some family $A \in \mathcal{U}^n(M)$ $(A \in \mathcal{C}^n(M))$ if and only if S is a dense $G_{\delta\sigma}$-subset. Moreover, for $n \geqslant 2$, if the mentioned family exists, it can be chosen from the class $\mathcal{Q}^n(M)$.*

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