On Nonpower-Law Behavior of Blow-up Solutions to Emden–Fowler Type Higher-Order Differential Equations

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1 Introduction

For the equation

$$y^{(n)} = p_0|y|^k \operatorname{sgn} y, \quad n \geq 2, \quad k > 1, \quad p_0 > 0, \tag{1.1}$$

we study blow-up solutions, i.e. those with $\lim\limits_{x \to x^* - 0} y(x) = \infty$.

The origin of the considered problem is described in [8, problem 16.4], and [6]. It was earlier proved for sufficiently large n (see [9]), for $n = 12$ (see [7]), for $n = 13, 14$ (see [4]), and for $n = 15$ (see [11]), that there exists $k = k(n) > 1$ such that equation (1.1) has a solution with nonpower-law behavior, namely,

$$y(x) = (x^* - x)^{-\alpha} h(\log(x^* - x)), \quad x \to x^* - 0, \tag{1.2}$$

where h is a positive periodic non-constant function on $\mathbb{R}$. Now we prove this result for arbitrary $n \geq 12$.

Note that it was also proved for $n = 2$ (see [8]) and for $n = 3, 4$ [1], that all blow-up solutions have power-law asymptotic behavior:

$$y(x) = C(x^* - x)^{-\alpha}(1 + o(1)), \quad x \to x^* - 0, \tag{1.3}$$

with

$$\alpha = \frac{n}{k-1}\,, \quad C = \Big(\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{p_0}\Big)^{\frac{1}{k-1}}. \tag{1.4}$$

Existence of a solution satisfying (1.3) was proved for arbitrary $n \geq 2$. For $2 \leq n \leq 11$ an $(n-1)$-parametric family of such solutions to equation (1.1) was proved to exist (see [1,2], [3, Ch. I(5.1)]). It was proved that for slightly superlinear equations of arbitrary order $n \geq 5$ all blow-up solutions have power-law asymptotic behavior (see [5]).

2 The main result

In this section, a result on existence of solutions with non-power behavior is formulated for equation (1.1) with $n \geq 12$.

Theorem 2.1. *For $n \geq 12$ there exists $k > 1$ such that equation* (1.1) *has a solution $y(x)$ with*

$$y^{(j)}(x) = (x^* - x)^{-\alpha - j} h_j(\log(x^* - x)), \quad j = 0, 1, \dots, n-1,$$

where α is defined by (1.4) *and h_j are periodic positive non-constant functions on $\mathbb{R}$.*

3 Proof of the main result

To prove the main result we transform equation (1.1) into the dynamical system and use a version of the Hopf Bifurcation theorem (see [10]).

3.1 Transformation of equation (1.1)

Equation (1.1) can be transformed into a dynamical system (see [1] or [3, Ch. I(5.1)]), by using the substitution

$$x^* - x = e^{-t}, \quad y = (C+v)\,e^{\alpha t}, \tag{3.1}$$

where C and α are defined by (1.4). The derivatives $y^{(j)}$, $j = 0, 1, \dots, n-1$, become

$$e^{(\alpha+j)t} \cdot L_j(v, v', \dots, v^{(j)}),$$

where $v^{(j)} = \frac{d^j v}{dt^j}$, and L_j is a linear function with

$$L_j(0, 0, \dots, 0) = C\alpha(\alpha+1)\cdots(\alpha+j-1) \neq 0$$

and the coefficient of $v^{(j)}$ is equal to 1.

Thus (1.1) is transformed into

$$e^{(\alpha+n)t} \cdot L_n(v, v', \dots, v^{(n)}) = p_0(C+v)^k e^{\alpha k t}, \tag{3.2}$$

$$v^{(n)} = p_0(C+v)^k - p_0 C^k - \sum_{j=0}^{n-1} a_j v^{(j)}, \tag{3.3}$$

where a_j, $j = 1, \dots, n$, are the coefficients of $v^{(j)}$ in the linear function L_n, and are $(n-j)$-degree polynomial functions in α. Equation (3.3) can be written as

$$v^{(n)} = kC^{k-1} p_0 v - \sum_{j=0}^{n-1} a_j v^{(j)} + f(v), \tag{3.4}$$

where

$$f(v) = p_0\big((C+v)^k - C^k - kC^{k-1}v\big) = O(v^2),$$
$$f'(v) = O(v) \ \text{ as } \ v \to 0,$$

Suppose $V = (V_0, \dots, V_{n-1})$ is the vector with coordinates $V_j = v^{(j)}$, $j = 0, \dots, n-1$. Then equation (3.4) can be written as

$$\frac{dV}{dt} = AV + F(V), \tag{3.5}$$

where A is a constant $n \times n$ matrix, namely,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ -\widetilde{a}_0 & -a_1 & -a_2 & -a_3 & \dots & -a_{n-1} \end{pmatrix}$$

with

$$\widetilde{a}_0 = a_0 - kc^{k-1}p_0 = a_0 - k\alpha(\alpha+1)\cdots(\alpha+n-1) = a_0 - (\alpha+1)\cdots(\alpha+n-1)(\alpha+n) \quad (3.6)$$

and eigenvalues satisfying the equation

$$\begin{aligned} 0 = \det(A-\lambda E) &= (-1)^{n+1}(-\widetilde{a}_0 - a_1\lambda - \cdots - a_{n-1}\lambda^{n-1} - \lambda^n) \\ &= (-1)^{n+1}\big((\alpha+1)(\alpha+2)\cdots(\alpha+n) - (\lambda+\alpha)\cdots(\lambda+\alpha+n-1)\big), \end{aligned} \quad (3.7)$$

which is equivalent to

$$\prod_{j=0}^{n-1}(\lambda+\alpha+j) = \prod_{j=0}^{n-1}(1+\alpha+j). \quad (3.8)$$

F in (3.5) is the vector function $F(V) = (0,\dots,0,F_{n-1}(V))$ and $F_{n-1}(V) = f(V_0)$.

3.2 Preliminary results

Theorem 3.1 (Modification of the Hopf Theorem [10]). *Consider an α-parameterized dynamical system $\dot{x} = f(x,\alpha)$ where $f : \mathbb{R}^{n+1} \mapsto \mathbb{R}^n$ is a C^r-function ($r \geq 3$) such that $f(0,\alpha) = 0$ for all $\alpha \in \mathbb{R}$. Suppose the Jacobian matrix $D_x f(0,\widetilde{\alpha}) \equiv A(\widetilde{\alpha})$ has $\pm i\beta$ as simple eigenvalues for some $\widetilde{\alpha} \in \mathbb{R}$. Let v and w be eigenvectors such that $Av = \beta iv$, $A^*w = \beta iw$, where A^* denotes the transpose conjugate matrix of the matrix A. Put*

$$\varphi \equiv \operatorname{Re}(e^{it}v), \quad \psi \equiv \operatorname{Re}(e^{it}w), \quad \Theta_j = \frac{1}{j!}\int_0^{2\pi}\Big(\frac{\partial^j(f_x)}{\partial\alpha^j}(0,\widetilde{\alpha})\varphi,\psi\Big)\,dt.$$

If $\Theta_c \neq 0$ for some odd number c, then $(0,\widetilde{\alpha})$ is a bifurcation point of periodic solutions of $\dot{x} = f(x,\alpha)$. More precisely, there exist continuous mappings $\varepsilon \mapsto \alpha(\varepsilon) \in \mathbf{R}$, $\varepsilon \mapsto T(\varepsilon) \in \mathbf{R}$, and $\varepsilon \mapsto b(\varepsilon) \in \mathbf{R}^n$ defined in a neighborhood of 0 and such that $\alpha(0) = \widetilde{\alpha}$, $T(0) = \frac{2\pi}{q}$, $b(0) = 0$, $b(\varepsilon) \neq 0$ for $\varepsilon \neq 0$, and the solutions to the problems $\dot{x} = f(x,\alpha(\varepsilon))$, $x(0) = b(\varepsilon)$ are $T(\varepsilon)$-periodic and non-constant.

To apply the Hopf Bifurcation theorem, we study equation (3.5) and the roots of the algebraic equation (3.8).

Lemma 3.1 ([4]). *For any integer $n \geq 12$ there exist $\alpha > 0$ and $q > 0$ such that*

$$\prod_{j=0}^{n-1}(qi+\alpha+j) = \prod_{j=0}^{n-1}(1+\alpha+j) \quad (3.9)$$

with $i^2 = -1$.

Lemma 3.2 ([4]). *For any $\alpha > 0$ and any integer $n > 1$ all roots $\lambda \in \mathbb{C}$ to equation (3.8) are simple.*

3.3 Proof of Theorem 2.1

We can obtain some useful formulas

$$\widetilde{a}_0 = \alpha(\alpha+1)\dots(\alpha+n-1) - (\alpha+1)\cdots(\alpha+n) = -n(\alpha+1)\dots(\alpha+n-1), \quad (3.10)$$

$$\frac{d^{n-1}(-\widetilde{a}_0)}{d\alpha^{n-1}} = n!, \quad \frac{d^{n-1}(-a_1)}{d\alpha^{n-1}} = -n!, \tag{3.11}$$

$$\frac{d^{n-2}(-\widetilde{a}_0)}{d\alpha^{n-2}} = n\Big((n-1)!\alpha + (n-2)!\,\frac{n(n-1)}{2}\Big) = \frac{(2\alpha+1)n!}{2}\,, \tag{3.12}$$

$$\frac{d^{n-1}(-a_2)}{d\alpha^{n-1}} = 0, \quad \frac{d^{n-2}(-a_2)}{d\alpha^{n-2}} = -(n-2)!\,\frac{n(n-1)}{2} = -\frac{n!}{2}\,. \tag{3.13}$$

By using (3.7), we can prove for n, α, q from Lemma 3.1 that the vector

$$v = (1, qi, -q^2, -q^3 i, q^4, \dots)$$

is an eigenvector of the matrix A corresponding to the eigenvalue qi. Consider also an eigenvector w of the matrix A^* corresponding to the eigenvalue qi, assuming its last coordinate to equal 1: $w = (\dots\dots, 1)$. Then

$$\varphi = \operatorname{Re}(e^{it}v) = \big(\cos t, -q\sin t, -q^2\cos t, q^3\sin t, q^4\cos t, \dots\big), \quad \psi = \operatorname{Re}(e^{it}w) = (\dots\dots, \cos t).$$

Using formulas (3.11)–(3.13), we obtain

$$\Theta_{n-1} = \frac{1}{(n-1)!}\int\limits_0^{2\pi} \left(\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ n! & -n! & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} \cos t \\ -q\sin t \\ \vdots \\ \vdots \end{pmatrix}, \begin{pmatrix} \vdots \\ \vdots \\ \vdots \\ \cos t \end{pmatrix} \right) dt$$

$$= \frac{1}{(n-1)!}\int\limits_0^{2\pi} n!\,(\cos^2 t + q\sin t\cos t)\,dt = \pi n \neq 0,$$

$$\Theta_{n-2} = \frac{1}{(n-2)!}\int\limits_0^{2\pi} \left(\begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \dfrac{(2\alpha+1)n!}{2} & \dfrac{d^{n-2}(-a_1)}{d\alpha^{n-2}} & -\dfrac{n!}{2} & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} \cos t \\ -q\sin t \\ -q^2\cos t \\ \vdots \end{pmatrix}, \begin{pmatrix} \vdots \\ \vdots \\ \vdots \\ \cos t \end{pmatrix} \right) dt$$

$$= \frac{\pi}{(n-2)!}\Big(\frac{(2\alpha+1)n!}{2} + \frac{q^2 n!}{2}\Big) = \frac{\pi n(n-1)}{2}(2\alpha+1+q^2) > 0.$$

So, if $n \geq 12$, then $\Theta_{n-1} > 0$, $\Theta_{n-2} > 0$ (since $\alpha > 0$), and either $n-1$ or $n-2$ is odd. Consequently, due to the above lemmas, all the conditions of Theorem 3.1 are fulfilled. Therefore, for any $n \geq 12$ there exists a family $\alpha_\varepsilon > 0$ such that equation (3.8) with $\alpha = \alpha_0$ has the imaginary roots $\lambda = \pm qi$ with q from Lemma 3.1 and, for sufficiently small ε, system (3.5) with $\alpha = \alpha_\varepsilon$ has an arbitrary small non-zero periodic solution $V_\varepsilon(t)$. In particular, the coordinate $V_{\varepsilon,0}(t) = v(t)$ of the vector $V_\varepsilon(t)$ is also a small periodic function with the same period. This function is non-zero, too. Otherwise, all $v^{(j)}$ and therefore $V_\varepsilon(t)$ itself should be zero. Then, taking into account (3.1), we obtain

$$y(x) = \big(C + v(-\ln(x^* - x))\big)(x^* - x)^{-\alpha}.$$

Put $h(s) = C + v(-s)$, which is a non-constant continuous periodic and positive for sufficiently small ε function, and obtain the required equality

$$y(x) = (x^* - x)^{-\alpha} h(\ln(x^* - x)).$$

In the similar way we obtain the related expressions for $y^{(j)}(x)$, $j = 0, 1, \dots, n-1$.

Theorem 2.1 is proved.

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