On the Well-Posedness Question of the Modified Cauchy Problem for Linear Systems of Impulsive Equations with Singularities

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Let $I \subset \mathbb{R}$ be an interval non-degenerate in the point, $b_0 = \sup I$ and

$$I_0 = I \setminus \{b_0\}.$$

Consider the linear system of impulsive equations with fixed points of impulses actions

$$\frac{dx}{dt} = P(t)x + q(t) \ \text{ for a.a. } \ t \in I_0 \setminus \{\tau_l\}_{l=1}^{+\infty}, \tag{1}$$

$$x(\tau_l+) - x(\tau_l-) = G_l x(\tau_l) + g_l \ \ (l = 1, 2, \dots), \tag{2}$$

where $P \in L_{loc}(I_0, \mathbb{R}^{n\times n})$, $q \in L_{loc}(I_0, \mathbb{R}^n)$, $G_l \in \mathbb{R}^{n\times n}$ $(l = 1, 2, \dots)$, $g_l \in \mathbb{R}^n$ $(l = 1, 2, \dots)$, $\tau_l \in I_0$ $(l = 1, 2, \dots)$, $\tau_i \neq \tau_j$ if $i \neq j$, and $\lim\limits_{l\to+\infty} \tau_l = b_0$.

Let $H = \operatorname{diag}(h_1, \dots, h_n) : I_0 \to \mathbb{R}^{n\times n}$ be a diagonal matrix-functions with continuous diagonal elements $h_k : I_0 \to]0, +\infty[$ $(k = 1, \dots, n)$.

We consider the problem of the well-posedness of solution $x : I_0 \to \mathbb{R}^n$ of the system $(1), (2)$, satisfying the modified Cauchy condition

$$\lim_{t\to b_0} (H^{-1}(t)x(t)) = 0. \tag{3}$$

Along with the system $(1), (2)$ consider the perturbed singular system

$$\frac{dx}{dt} = \widetilde{P}(t)x + \widetilde{q}(t) \ \text{ for a.a. } \ t \in I_0 \setminus \{\tau_l\}_{l=1}^{+\infty}, \tag{4}$$

$$x(\tau_l+) - x(\tau_l-) = \widetilde{G}_l x(\tau_l) + \widetilde{g}_l \ \ (l = 1, 2, \dots), \tag{5}$$

where $\widetilde{P} \in L_{loc}(I_0, \mathbb{R}^{n\times n})$, $\widetilde{q} \in L_{loc}(I_0, \mathbb{R}^n)$, $\widetilde{G}_l \in \mathbb{R}^{n\times n}$ $(l = 1, 2, \dots)$, $\widetilde{g}_l \in \mathbb{R}^n$ $(l = 1, 2, \dots)$.

In the paper, we investigate the question when the unique solvability of the problem $(1), (2); (3)$ guarantees the unique solvability of the problem $(4), (5); (3)$ and also nearness of its solutions in the definite sense if matrix-functions P and $\widetilde{P}$, G_l and $\widetilde{G}_l$ $(l = 1, 2, \dots)$, and vector-functions q and $\widetilde{q}$ and g_l and $\widetilde{g}_l$ $(l = 1, 2, \dots)$ are accordingly close to each other.

The analogous problem for systems (1) of ordinary differential equations with singularities are investigated in [2–4].

The singularity of system (1) is considered in the sense that the matrix P and vector q functions, in general, are not integrable at the point b. In general, the solution of the problem $(1), (2); (3)$

is not continuous at the point b and, therefore, it is not a solution in the classical sense. But its restriction on every interval from I_0 is a solution of the system (1), (2). In connection with this we give the example from [4].

Let $\alpha > 0$ and $\varepsilon \in]0, \alpha[$. Then the problem

$$\frac{dx}{dt} = -\frac{\alpha\, x}{t} + \varepsilon |t|^{\varepsilon-1-\alpha}, \quad \lim_{t\to 0}(t^\alpha x(t)) = 0$$

has the unique solution $x(t) = |t|^{\varepsilon-\alpha} \operatorname{sgn} t$. This function is not solution of the equation on the set $I = \mathbb{R}$, but its restrictions on $]-\infty, 0[$ and $]0, +\infty[$ are solutions of that.

We give sufficient conditions guaranteeing the well-posedness of the problem (1), (2); (3). The analogous results belong to I. Kiguradze [3, 4] for the modified Cauchy problem for systems of ordinary differential equations with singularities.

Some boundary value problems for linear impulsive systems with singularities are investigated in [1] (see, also references therein).

In the paper, the use will be made of the following notation and definitions.

$\mathbb{N}$ is the set of all natural numbers.

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$, $[a, b]$ and $]a, b[$ $(a, b \in \mathbb{R})$ are, respectively, closed and open intervals.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\| = \max\limits_{j=1,\dots,m} \sum\limits_{i=1}^{n} |x_{ij}|$.

If $X = (x_{ij})_{i,j=1}^{n,m} \in \mathbb{R}^{n\times m}$, then $|X| = (|x_{ij}|)_{i,j=1}^{n,m}$.

$\mathbb{R}_+^{n\times m} = \{(x_{ij})_{i,j=1}^{n,m} :\ x_{ij} \geq 0 \ (i = 1, \dots, n;\ j = 1, \dots, m)\}$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X; I_n is the identity $n \times n$-matrix.

The inequalities between the matrices are understood componentwise.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of the matrix-function $X : [a, b] \to \mathbb{R}^{n\times m}$ at the point t.

$\widetilde{C}([a, b], D)$, where $D \subset \mathbb{R}^{n\times m}$, is the set of all absolutely continuous matrix-functions $X : [a, b] \to D$.

$\widetilde{C}_{loc}(I_0 \setminus \{\tau_l\}_{l=1}^{+\infty}, D)$ is the set of all matrix-functions $X : I_{t_0} \to D$ whose restrictions to an arbitrary closed interval $[a, b]$ from $I_0 \setminus \{\tau_l\}_{l=1}^{+\infty}$ belong to $\widetilde{C}([a, b], D)$.

$L([a, b]; D)$ is the set of all integrable matrix-functions $X : [a, b] \to D$.

$L_{loc}(I_0; D)$ is the set of all matrix-functions $X : I_0 \to D$ whose restrictions to an arbitrary closed interval $[a, b]$ from I_0 belong to $L([a, b], D)$.

A vector-function $x \in \widetilde{C}_{loc}(I_0 \setminus \{\tau_l\}_{l=1}^{+\infty}, \mathbb{R}^n)$ is said to be a solution of the system (1), (2) if

$$x'(t) = P(t)x(t) + q(t) \ \text{ for a.a. } \ t \in I_{t_0} \setminus \{\tau_l\}_{l=1}^{+\infty}$$

and there exist one-sided limits $x(\tau_l-)$ and $x(\tau_l+)$ $(l = 1, 2, \dots)$ such that the equalities (2) hold.

We assume that

$$\det(I_n + G_l) \neq 0 \ \ (l = 1, 2, \dots).$$

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding nonsingular systems, i.e. for the case when $P \in L_{loc}(I, \mathbb{R}^{n\times n})$ and $q \in L_{loc}(I, \mathbb{R}^n)$.

Let $\mathcal{N}_{t0} = \{l \in \mathbb{N} :\ t \leq \tau_l < b\}$ and $I_0(\delta) = [b_0 - \delta, b_0[\cap I_0$ for every $\delta > 0$.

Definition. The problem (1), (2); (3) is said to be H-well-posed if it has the unique solution x and for every $\varepsilon > 0$ there exists $\eta > 0$ such that the problem (4), (5); (3) has the unique solution $\widetilde{x}$ and the estimate

$$\big\|H(t)(x(t) - \widetilde{x}(t))\big\| < \varepsilon \ \text{ for } \ t \in I$$

holds for every $\widetilde{P} \in L_{loc}(I_0, \mathbb{R}^{n\times n})$, $\widetilde{q} \in L_{loc}(I_0, \mathbb{R}^n)$, $\widetilde{G}_l \in \mathbb{R}^{n\times n}$ $(l = 1, 2, \dots)$, $\widetilde{g}_l \in \mathbb{R}^n$ $(l = 1, 2, \dots)$ such that $\det(I_n + \widetilde{G}_l) \neq 0$ $(l = 1, 2, \dots)$,

$$\bigg\|\int\limits_t^{b-} H^{-1}(s)|\widetilde{P}(s) - P(s)|H(s)\,ds\bigg\| + \bigg\|\sum_{l\in\mathcal{N}_{t0}} H^{-1}(\tau_l)|\widetilde{G}_l - G_l|H(\tau_l)\bigg\| < \eta \ \text{ for } \ t \in I_0(\delta)$$

and

$$\bigg\|\int\limits_t^{b-} H^{-1}(s)|\widetilde{q}(s) - q(s)|\,ds\bigg\| + \bigg\|\sum_{l\in\mathcal{N}_{t0}} H^{-1}(\tau_l)|\widetilde{g}_l - g_l|\,\bigg\| < \eta \ \text{ for } \ t \in I_0(\delta).$$

Let $P_0 \in L_{loc}(I_{t_0}, \mathbb{R}^{n\times n})$ and $G_{0l} \in \mathbb{R}^{n\times n}$ $(l = 1, 2, \dots)$. Then a matrix-function $C_0 : I_0 \times I_0 \to \mathbb{R}^{n\times n}$ is said to be the Cauchy matrix of the homogeneous impulsive system

$$\frac{dx}{dt} = P_0(t)x, \tag{6}$$

$$x(\tau_l+) - x(\tau_l-) = G_{0l}x(\tau_l) \ \ (l = 1, 2, \dots), \tag{7}$$

if, for every interval $J \subset I_0$ and $\tau \in J$, the restriction of $C_0(\,\cdot\,, \tau) : I_0 \to \mathbb{R}^{n\times n}$ on J is the fundamental matrix of the system (6), (7) satisfying the condition $C_0(\tau, \tau) = I_n$. Therefore, C_0 is the Cauchy matrix of (6), (7) if and only if the restriction of C_0 on $J \times J$, for every interval $J \subset I_0$, is the Cauchy matrix of the system in the sense of definition given in [5].

Theorem. *Let there exist a matrix-function $P_0 \in L_{loc}(I_{t_0}, \mathbb{R}^{n\times n})$ and constant matrices $G_l \in \mathbb{R}^{n\times n}$ $(l = 1, 2, \dots)$ and $B_0, B \in \mathbb{R}^{n\times n}_+$ such that*

$$\det(I_n + G_{0l}) \neq 0 \ \ (l = 1, 2, \dots),$$
$$r(B) < 1,$$

and the estimates

$$|C_0(t, \tau)| \le H(t)\,B_0\,H^{-1}(\tau) \ \ \textit{for} \ \ b - \delta \le t \le \tau < b, \ \ \tau \neq \tau_l \ \ (l = 1, 2, \dots),$$
$$\big|C_0(t, \tau_l)G_{0l}(I_n + G_{0l})^{-1}\big| \le H(t)B_0H^{-1}(\tau_l) \ \ \textit{for} \ \ b - \delta \le t \le \tau_l < b \ \ (l = 1, 2, \dots)$$

and

$$\int\limits_t^{b-} \big|C_0(t, \tau)(P(\tau) - P_0(\tau))\big|H(\tau)\,d\tau + \sum_{l\in\mathcal{N}_{t0}} \big|C_0(t, \tau_l)G_{0l}(I_n + G_{0l})^{-1}\big|\,|G_l - G_{0l}|H(\tau_l) \le H(t)B \ \ \textit{for} \ \ t \in I_0(\delta)$$

hold for some $\delta > 0$, where C_0 is the Cauchy matrix of the system (5), (6). *Let, moreover,*

$$\lim_{t\to b}\Bigg(\bigg\|\int\limits_{t_0}^{t} H^{-1}(\tau)|C_0(t, \tau)|\,|q(\tau)|\,d\tau\bigg\| + \bigg\|\sum_{l\in\mathcal{N}_{t0}} H^{-1}(\tau_l)\big|C_0(t, \tau_l)G_{0l}(I_n + G_{0l})^{-1}\big|\,|g_l|\,\bigg\|\Bigg) = 0.$$

Then the problem (1), (2); (3) *is H-well-posed.*

References

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