Non-Instantaneous Impulsive Differential Equations with Finite State Dependent Delay and Ulam-Type Stability

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Abstract

We consider an initial value problem (IVP) for a nonlinear system of non-instantaneous impulsive differential equations with state dependent delay (NIDDE) and Ulam-type stability is studied.

1 Statement of the problem

Let the points $t_i, s_i \in [0,T]$: $s_0 = 0$, $t_{k+1} = T$, $0 < t_i < s_i < t_{i+1}$, $i = 1, 2, \dots, k$ be given. Consider the space $PC_0 = C([-r,0], E)$ endowed with the norm $\|y\|_{PC_0} = \sup\limits_{t\in[-r,0]} \{\|y(t)\|_E : \ y \in PC_0\}$; here E is a Banach space.

The intervals (s_i, t_{i+1}), $i = 0, 1, 2, \dots, k$ will be the intervals on which the fractional differential equation will be given and the intervals (t_i, s_i), $i = 1, 2, \dots, k$ will be called impulsive intervals and on these intervals impulsive conditions are given.

Consider the IVP for the NIDDE

$$\begin{aligned}
&x'(t) = f(t, x_{\rho(t,x_t)}) \ \text{ for } \ t \in (s_i, t_{i+1}], \ \ i = 0, 1, 2, \dots, k,\\
&x(t) = g_i(t, x(t_i)), \ \ t \in (t_i, s_i], \ \ i = 1, 2, \dots, k,\\
&x(t) = \phi(t) \ \text{ for } \ t \in [-r, 0],
\end{aligned} \tag{1.1}$$

where the functions $f : [0,T] \times PC_0 \to E$; $\rho : [0,T] \times PC_0 \to [0,T]$, $\phi : [-r,0] \to E$; $g_i : [t_i, s_i] \times E \to E$, $i = 1, 2, \dots, k$. Here for any $t \in [0,T]$ the notation $x_t(s) = x(t+s)$, $s \in [-r,0]$ is used, i.e. x_t represents the history of the state $x(t)$ from time $t-r$ up to the present time t. Note that for any $t \in [0,T]$ we let $y_{\rho(t,x_t)}(s) = x(\rho(t, x(t+s)) + s)$, $s \in [-r,0]$, i.e. the function ρ determines the state-dependent delay.

Remark 1.1. Note in the special case $\rho(t,x)\equiv t$ problem (1.1) reduces to an IVP for a delay non-instantaneous impulsive differential equation.

Let $\mathcal{PC}$ be the Banach space of all functions $y:[-r,T]\to E$ which are continuous on $[0,T]$ except for the points $t_i\in(0,T)$ at which $y(t_i+)=\lim\limits_{t\downarrow t_i}y(t)$ and $y(t_i-)=y(t_i)=\lim\limits_{t\uparrow t_i}y(t)$ exist and it is endowed with the norm $\|y\|_{\mathcal{PC}}=\sup\limits_{t\in[-r,T]}\{\|y(t)\|_E:\ y\in\mathcal{PC}\}$.

We consider the assumptions:

A1. The function $f\in C\big(\bigcup\limits_{i=0}^{k}[s_i,t_{i+1}]\times E,E\big)$.

A2. The function $\phi\in PC_0$.

A3. The function $\rho\in C\big(\bigcup\limits_{i=0}^{k}[s_i,t_{i+1}]\times E,[0,T]\big)$ is such that for any $t\in\bigcup\limits_{i=0}^{k}[s_i,t_{i+1}]$ and any function $u\in PC_0$ the inequality $\rho(t,u)\le t$ holds.

A4. The functions $g_i\in C([t_i,s_i]\times E,E)$, $i=1,2,\dots,k$.

Definition 1.1. The function $x\in\mathcal{PC}$ is a solution of the IVP (1.1) iff it satisfies the following integral-algebraic equation

$$x(t)=\begin{cases}\phi(t), & t\in[-r,0],\\ \phi(0)+\displaystyle\int_0^t f(s,x_{\rho(s,x_s)})\,ds, & t\in(0,t_1],\\ g_i(t,x(t_i)), & t\in(t_i,s_i],\quad i=1,2,\dots,k,\\ g_i(s_i,x(t_i))+\displaystyle\int_{s_i}^t f(s,x_{\rho(s,x_s)})\,ds, & t\in(s_i,t_{i+1}],\quad i=1,2,\dots,k.\end{cases}\tag{1.2}$$

2 Ulam types stability

Let $\varepsilon>0$, $\Psi\ge0$ and $\Phi\in C\big(\bigcup\limits_{i=1}^{k}[s_i,t_{i+1}],[0,\infty)\big)$ be nondecreasing. We consider the following inequalities:

$$\begin{gathered}\big\|y'(t)-f(t,y_{\rho(t,y_t)})\big\|_E\le\varepsilon\ \text{ for }\ t\in(s_i,t_{i+1}],\quad i=0,1,2,\dots,k,\\ \big\|y(t)-g_i(t,y(t_i))\big\|_E\le\varepsilon,\ \ t\in(t_i,s_i],\quad i=1,2,\dots,k,\end{gathered}\tag{2.1}$$

and

$$\begin{gathered}\big\|y'(t)-f(t,y_{\rho(t,y_t)})\big\|_E\le\Phi(t)\ \text{ for }\ t\in(s_i,t_{i+1}],\quad i=0,1,2,\dots,k,\\ \big\|y(t)-g_i(t,y(t_i))\big\|_E\le\Psi,\ \ t\in(t_i,s_i],\quad i=1,2,\dots,k,\end{gathered}\tag{2.2}$$

and

$$\begin{gathered}\big\|y'(t)-f(t,y_{\rho(t,y_t)})\big\|_E\le\varepsilon\Phi(t)\ \text{ for }\ t\in(s_i,t_{i+1}],\quad i=0,1,2,\dots,k,\\ \big\|y(t)-g_i(t,y(t_i))\big\|_E\le\varepsilon\Psi,\ \ t\in(t_i,s_i],\quad i=1,2,\dots,k.\end{gathered}\tag{2.3}$$

The inequalities (2.1)–(2.3) have connections with the definitions of Ulam–Hyers stability, Ulam–Hyers–Rassias stability with respect to Φ, Ψ and generalized Ulam–Hyers–Rassias stability, respectively (for detailed definitions see, for example [2]).

Lemma 2.1. *Let assumptions* A1, A3, A4 *be satisfied.*

- *If $y \in \mathcal{PC}$ is a solution of inequalities* (2.1)*, then it satisfies the following integral-algebraic inequalities*

$$\begin{cases} \Big\| y(t) - \phi(0) - \int\limits_0^t f(s, y_{\rho(s,y_s)})\,ds \Big\|_E \le \varepsilon t, & t \in (0, t_1], \\ \big\| y(t) - g_i(t, y(t_i)) \big\|_E \le \varepsilon, & t \in (t_i, s_i], \;\; i = 1, 2, \dots, k, \\ \Big\| y(t) - g_i(s_i, y(t_i)) - \int\limits_{s_i}^t f(s, y_{\rho(s,y_s)})\,ds \Big\|_E \le \varepsilon + \varepsilon(t - s_i), & t \in (s_i, t_{i+1}], \;\; k = 1, 2, \dots, k. \end{cases}$$

- *If $y \in \mathcal{PC}$ is a solution of inequalities* (2.2)*, then it satisfies the following integral-algebraic inequalities*

$$\begin{cases} \Big\| y(t) - \phi(0) - \int\limits_0^t f(s, y_{\rho(s,y_s)})\,ds \Big\|_E \le \int\limits_0^t \Phi(s)\,ds, & t \in (0, t_1], \\ \big\| y(t) - g_i(t, y(t_i)) \big\|_E \le \Psi, & t \in (t_i, s_i], \;\; i = 1, 2, \dots, k, \\ \Big\| y(t) - g_i(s_i, y(t_i)) - \int\limits_{s_i}^t |f(s, y_{\rho(s,y_s)})|\,ds \Big\|_E \le \Psi + \int\limits_{s_i}^t \Phi(s)\,ds, & t \in (s_i, t_{i+1}], \;\; i = 1, 2, \dots, k. \end{cases}$$

Remark 2.1. We have a similar result for the inequality (2.3).

Next we discuss the existence of the solution of (1.1), given by Definition 1.1, using the Banach contraction principle.

Theorem 2.1 (*Existence result*)**.** *Let the following conditions be satisfied:*

1. *Assumption* A1 *is satisfied and there exists a constant $L_f > 0$ such that for any $t \in \bigcup\limits_{i=1}^k [s_i, t_{i+1}]$ and any functions $u, v \in \mathcal{PC}$ the inequality*

$$\big\| f(t, u_{\rho(t,u_t)}) - f(t, v_{\rho(t,v_t)}) \big\|_E \le L_f \| u_{\rho(t,u_t)} - v_{\rho(t,v_t)} \|_{PC_0}$$

holds.

2. *Assumption* A4 *is satisfied and there exist constants $L_{g_i} > 0$, $i = 1, 2, \dots, k$, such that*

$$\| g_i(t, x) - g_i(t, y) \|_E \le L_{g_i} \| x - y \|_E, \;\; t \in [t_i, s_i], \;\; x, y \in E, \;\; i = 1, 2, \dots, k.$$

3. *Assumptions* A2*,* A3 *are satisfied.*

4. *The inequality $\gamma = \max\limits_{i=1,2,\dots,k} L_{g_i} + \eta L_f < 1$ holds, where $\eta = \max\{t_{i+1} - s_i, \; i = 0, 1, \dots, k\}$.*

Then the initial value problem (1.1) *has a unique solution $x \in \mathcal{PC}$ as defined in Definition* 1.1*.*

Theorem 2.2 (*Stability results*)**.** *Let the conditions of Theorem* 2.1 *be satisfied.*

(i) *Assume for any $\varepsilon > 0$ inequality* (2.1) *has at least one solution $y_\varepsilon \in \mathcal{PC}$. Then problem* (1.1) *is Ulam–Hyers stable, i.e.*

$$\|x(t) - y_\varepsilon(t)\|_E < c_{f,g_i}\varepsilon, \ \ t \in [0,T]$$

with

$$c_{f,g_i} = 1 + (1+\eta)\sum_{j=1}^{k-1}\Big(\prod_{m=0}^{j-1} L_{g_{k-m}}\Big)e^{jL_f\eta} + \Big(\prod_{j=1}^{k} L_{g_j}\Big)\eta e^{(k+1)L_f\eta},$$

where x is the solution of (1.1).

(ii) *Suppose there exist constants $\Psi \geq 0$, $\Lambda_\Phi > 0$ and a function $\Phi \in C\big(\bigcup\limits_{i=1}^{k}[s_i,t_{i+1}],[0,\infty)\big)$ such that for any $t \in [s_i,t_{i+1}]$, $i = 0,1,2,\dots,k$ inequality $\int\limits_{s_i}^{t}\Phi(s)\,ds \leq \Lambda_\Phi\Phi(t)$ holds and for any $\varepsilon > 0$ inequality* (2.3) *has at least one solution $y_\varepsilon(t) \in \mathcal{PC}$. Then problem* (1.1) *is generalized Ulam–Hyers–Rassias stable with respect to Φ, Ψ.*

(iii) *Assume there exist constants $\Psi \geq 0$, $\Lambda_\Phi > 0$ and a function $\Phi \in C\big(\bigcup\limits_{i=1}^{k}[s_i,t_{i+1}],[0,\infty)\big)$ such that for any $t \in [s_i,t_{i+1}]$, $i = 0,1,2,\dots,k$ inequality $\int\limits_{s_i}^{t}\Phi(s)\,ds \leq \Lambda_\Phi\Phi(t)$ holds and inequality* (2.2) *has at least one solution $y \in \mathcal{PC}$. Then problem* (1.1) *is Ulam–Hyers–Rassias stable with respect to Φ, Ψ, i.e. $\|x(t) - y(t)\|_E < c_{f,g_i}(\Psi + \Phi(t))$, $t \in [0,T]$ with*

$$C = \max\{1,\Lambda_\Phi\}, \quad c_{f,g_i} = Ce^{L_f\eta}\Big(1 + \sum_{i=1}^{k}\prod_{m=0}^{i-1}(L_{g_{k-m}}e^{L_f\eta})\Big),$$

where x is the solution of (1.1).

Remark 2.2. Ulam stability properties of ordinary differential equations were studied in [2], for impulsive differential equations without any type of delays see [3] and for impulsive differential equations with variable delays see [4].

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