The Shift Invariance of Time Scales and Applications

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Abstract Recently Wang, Agarwal and O'Regan studied the problem of the shift invariance of time scales. This contributes to investigating functions defined by shifts, for example periodic functions, almost periodic functions and almost automorphic functions, etc. Moreover the related theory of dynamic equations was established. In addition, the shift invariance of time scales was employed to construct delay functions effectively in delay dynamic equations on time scales. As an extension, the almost translation invariance of time scales has been presented.

1 A matched space and the shift invariance of time scales

Recently, to consider the problem of the shift invariance of time scales, the authors introduced the concept of matched space for time scales and applied it to propose new concepts of almost periodic functions and almost automorphic functions (see [7]). These concepts are effective not only on periodic time scales but also are valid on irregular time scales such as quantum-like time scales, e.g. $\pm\overline{q^{\mathbb{Z}}}$, $\overline{(-q)^{\mathbb{Z}}}$, $\overline{-q^{\mathbb{Z}} \cup q^{\mathbb{Z}}}$ and others such as $\pm\mathbb{N}^{\frac{1}{2}}$ and $\pm\mathbb{N}^2$, etc. Under a matched space for the time scale, a new almost periodic theory for dynamic equations was established (see [10]). To investigate periodic solutions to dynamic equations on a quantum time scale, Adıvar proposed a new concept of periodic time scales under shift operators in 2013, and this approach enables one to investigate this periodic notion on time scales (see [1]). In 2017, using the idea of shift operators established in [1], the authors introduced the concept of piecewise almost periodic functions which includes q-difference equations on a quantum-like time scale under a stochastic background and they used it to study almost periodic solutions to stochastic impulsive dynamic delay models (see [8, 11]).

To develop an effective tool to consider the shift invariance of time scales the authors in [7] used the algebraic structure of an Abelian group and introduced the concept of a matched space for time scales and they constructed the algebraic structure of matched spaces to solve the problem of closedness of time scales under shifts (including translational and non-translational shifts). The concept of periodic time scales under matched spaces includes the definition of periodic time scales proposed by Adıvar. Also functions defined by shifts can be introduced on a larger group of time scales and the shift invariance of time scales can be guaranteed under the matched space.

To introduce the concept of matched space for time scales, the following algebraic structure for a pair $(\Pi^*, \widetilde{\delta})$ is needed.

Definition 1.1 ([7])**.** Let Π^* be a subset of $\mathbb{R}$ together with an operation $\widetilde{\delta}$ and a pair $(\Pi^*, \widetilde{\delta})$ be an Abelian group and $\widetilde{\delta}$ be increasing with respect to its second argument, i.e., Π^* and $\widetilde{\delta}$ satisfy the following conditions:

(1) Π^* is closed with respect to an operation $\widetilde{\delta}$, i.e., for any $\tau_1, \tau_2 \in \Pi^*$, we have $\widetilde{\delta}(\tau_1, \tau_2) \in \Pi^*$.

(2) For any $\tau \in \Pi^*$, there exists an identity element $e_{\Pi^*} \in \Pi^*$ such that $\widetilde{\delta}(e_{\Pi^*}, \tau) = \tau$.

(3) For all $\tau_1, \tau_2, \tau_3 \in \Pi^*$, $\widetilde{\delta}(\tau_1, \widetilde{\delta}(\tau_2, \tau_3)) = \widetilde{\delta}(\widetilde{\delta}(\tau_1, \tau_2), \tau_3)$ and $\widetilde{\delta}(\tau_1, \tau_2) = \widetilde{\delta}(\tau_2, \tau_1)$.

(4) For each $\tau \in \Pi^*$, there exists an element $\tau^{-1} \in \Pi^*$ such that $\widetilde{\delta}(\tau, \tau^{-1}) = \widetilde{\delta}(\tau^{-1}, \tau) = e_{\Pi^*}$, where e_{Π^*} is the identity element in Π^*.

(5) If $\tau_1 > \tau_2$, then $\widetilde{\delta}(\,\cdot\,, \tau_1) > \widetilde{\delta}(\,\cdot\,, \tau_2)$.

A subset S of $\mathbb{R}$ is called relatively dense with respect to the pair $(\Pi^*, \widetilde{\delta})$ if there exists a number $L \in \Pi^*$ and $L > e_{\Pi^*}$ such that $[a, \widetilde{\delta}(a, L)]_{\Pi^*} \cap S \neq \varnothing$ for all $a \in \Pi^*$. The number $|L|$ is called the inclusion length with respect to the group $(\Pi^*, \widetilde{\delta})$.

According to Definition 1.1, one can introduce the definition of a relatively dense set with respect to the group $(\Pi^*, \widetilde{\delta})$, where Π^* is a subset of $\mathbb{R}$ together with an operation $\widetilde{\delta}$.

Definition 1.2 ([7])**.** A subset S of $\mathbb{R}$ is called relatively dense with respect to the pair $(\Pi^*, \widetilde{\delta})$ if there exists a number $L \in \Pi^*$ and $L > e_{\Pi^*}$ such that $[a, \widetilde{\delta}(a, L)]_{\Pi^*} \cap S \neq \varnothing$ for all $a \in \Pi^*$. The number $|L|$ is called the inclusion length with respect to the group $(\Pi^*, \widetilde{\delta})$.

Definition 1.3 ([7])**.** Let the pair $(\Pi^*, \widetilde{\delta})$ be an Abelian group and Π^*, $\mathbb{T}^*$ be the largest subsets of the time scales Π and $\mathbb{T}$, respectively. Further, let Π be the adjoint set of $\mathbb{T}$ and F the adjoint mapping between $\mathbb{T}$ and Π. The operator $\delta : \Pi^* \times \mathbb{T}^* \to \mathbb{T}^*$ satisfies the following properties:

(P_1) (Monotonicity) The function δ is strictly increasing with respect to its all arguments, i.e., if

$$(T_0, t),\ (T_0, u) \in \mathcal{D}_\delta := \big\{(s, t) \in \Pi^* \times \mathbb{T}^* :\ \delta(s, t) \in \mathbb{T}^*\big\},$$

then $t < u$ implies $\delta(T_0, t) < \delta(T_0, u)$; if (T_1, u), $(T_2, u) \in \mathcal{D}_\delta$ with $T_1 < T_2$, then $\delta(T_1, u) < \delta(T_2, u)$.

(P_2) (Existence of inverse elements) The operator δ has the inverse operator $\delta^{-1} : \Pi^* \times \mathbb{T}^* \to \mathbb{T}^*$ and $\delta^{-1}(\tau, t) = \delta(\tau^{-1}, t)$, where $\tau^{-1} \in \Pi^*$ is the inverse element of τ.

(P_3) (Existence of identity element) $e_{\Pi^*} \in \Pi^*$ and $\delta(e_{\Pi^*}, t) = t$ for any $t \in \mathbb{T}^*$, where e_{Π^*} is the identity element in Π^*.

(P_4) (Bridge condition) For any $\tau_1, \tau_2 \in \Pi^*$ and $t \in \mathbb{T}^*$, $\delta(\widetilde{\delta}(\tau_1, \tau_2), t) = \delta(\tau_1, \delta(\tau_2, t)) = \delta(\tau_2, \delta(\tau_1, t))$.

Then the operator $\delta(s, t)$ associated with $e_{\Pi^*} \in \Pi^*$ is said to be a shift operator on the set $\mathbb{T}^*$. The variable $s \in \Pi^*$ in δ is called the shift size. The value $\delta(s, t)$ in $\mathbb{T}^*$ indicates s units shift of the term $t \in \mathbb{T}^*$. The set $\mathcal{D}_\delta$ is the domain of the shift operator δ.

Definition 1.4. Let the pair $(\Pi^*, \widetilde{\delta})$ be an Abelian group, and Π^*, $\mathbb{T}^*$ be the largest subsets of the time scales Π and $\mathbb{T}$, respectively. Further, let Π be an adjoint set of $\mathbb{T}$ and F the adjoint mapping between $\mathbb{T}$ and Π. If there exists the shift operator δ satisfying Definition 1.3, then we say the group $(\mathbb{T}, \Pi, F, \delta)$ is a matched space for the time scale $\mathbb{T}$.

Using the algebraic structure of matched spaces, we introduce the following new concept of periodic time scales.

Definition 1.5. A time scale $\mathbb{T}$ is called a periodic time scale (or bi-direction shift invariant time scale) under a matched space $(\mathbb{T}, \Pi, F, \delta)$ if

$$\widetilde{\Pi} := \big\{\tau \in \Pi^* : \ (\tau^{\pm 1}, t) \in \mathcal{D}_\delta, \ \forall\, t \in \mathbb{T}^*\big\} \notin \big\{\{e_{\Pi^*}\}, \varnothing\big\}. \tag{1.1}$$

The shift invariance of time scales attached with shift directions is also considered in [7]. In this case, the pair $(\Pi^*, \widetilde{\delta})$ in Definition 1.1 is not an Abelian group but a semigroup with a direction, which indicates that the shift direction has an impact on the shift closedness of time scales (see Definition 2.8, Definition 2.10 from [7]). In [6] the authors noted that that the translation direction should be taken into account when one considers the translation invariance of time scales and the authors introduced the concept of periodic time scales attached with translation direction (i.e., oriented-direction translation invariant time scales). During 2015–2017, Wang, Agarwal and O'Regan considered the local translation invariance of time scales attached with translation direction and introduced the concept of changing-periodic time scales and established a composition theorem of time scales to divide an arbitrary time scale into a countable union of translation invariant time scales which are with translation directions (see [2, 4]). Therefore, shift direction is also another factor that should be considered when discussing the shift invariance of time scales.

2 Almost periodic and almost automorphic functions

Under the matched space of time scales, some new concepts of almost periodic functions and almost automorphic functions were introduced and a related theory with applications to dynamic equations was established in the literature (see [7, 10]).

Now, we assume that $(\mathbb{T}, \Pi, F, \delta)$ is a bi-direction matched space, then all the elements from Π^* have the corresponding inverse elements in Π^*.

Definition 2.1. Let $\mathbb{T}$ be a periodic time scale under the matched space $(\mathbb{T}, \Pi, F, \delta)$. A function $f \in C(\mathbb{T} \times D, \mathbb{X})$ is called an almost periodic function with shift operators in $t \in \mathbb{T}$ uniformly for $x \in D$ if the ε-shift set of f

$$E\{\varepsilon, f, S\} = \Big\{\tau \in \widetilde{\Pi} : \ \big\|f(\delta_{\tau^{\pm 1}}(t), x) - f(t, x)\big\| < \varepsilon \ \text{ for all } t \in \mathbb{T}^* \text{ and } x \in S\Big\}$$

is a relatively dense set with respect to the pair $(\Pi^*, \widetilde{\delta})$ for all $\varepsilon > 0$ and for each compact subset S of D; that is, for any given $\varepsilon > 0$ and each compact subset S of D, there exists a constant $l(\varepsilon, S) > 0$ such that each interval of length $l(\varepsilon, S)$ contains a $\tau(\varepsilon, S) \in E\{\varepsilon, f, S\}$ such that

$$\big\|f(\delta_{\tau^{\pm 1}}(t), x) - f(t, x)\big\| < \varepsilon \ \text{ for all } t \in \mathbb{T}^* \text{ and } x \in S.$$

Now τ is called the ε-shift number of f and $l(\varepsilon, S)$ is called the inclusion length of $E\{\varepsilon, f, S\}$.

Definition 2.2. Let $\mathbb{T}$ be a periodic time scale under the matched space $(\mathbb{T}, \Pi, F, \delta)$, the shift $\delta_{\tau^{\pm 1}}(t)$ is Δ-differentiable with rd-continuous bounded derivatives $\delta^{\Delta}_{\tau^{\pm 1}}(t) := \delta^{\Delta}(\tau^{\pm 1}, t)$ for all

$t \in \mathbb{T}^*$. A function $f \in C(\mathbb{T} \times D, \mathbb{X})$ is called an almost periodic function with shift operators in $t \in \mathbb{T}$ uniformly for $x \in D$ if the ε-shift set of f

$$E\{\varepsilon, f, S\} = \Big\{\tau \in \widetilde{\Pi} : \ \big\| f(\delta_{\tau^{\pm 1}}(t), x)\delta^{\Delta}_{\tau^{\pm 1}}(t) - f(t,x)\big\| < \varepsilon \ \text{ for all } t \in \mathbb{T}^* \text{ and } x \in S\Big\}$$

is a relatively dense set with respect to the pair $(\Pi^*, \widetilde{\delta})$ for all $\varepsilon > 0$ and for each compact subset S of D; that is, for any given $\varepsilon > 0$ and each compact subset S of D, there exists a constant $l(\varepsilon, S) > 0$ such that each interval of length $l(\varepsilon, S)$ contains a $\tau(\varepsilon, S) \in E\{\varepsilon, f, S\}$ such that

$$\big\| f(\delta_{\tau^{\pm 1}}(t), x)\delta^{\Delta}_{\tau^{\pm 1}}(t) - f(t,x)\big\| < \varepsilon \ \text{ for all } t \in \mathbb{T}^* \text{ and } x \in S.$$

Now τ is called the ε-shift number of f and $l(\varepsilon, S)$ is called the inclusion length of $E\{\varepsilon, f, S\}$.

Definition 2.3. Let $\mathbb{T}$ be a periodic time scale under the matched space $(\mathbb{T}, \Pi, F, \delta)$.

(i) Let $f : \mathbb{T} \to \mathbb{X}$ be a bounded continuous function. We say that f is almost automorphic if from every sequence $\{s_n\} \subset \widetilde{\Pi}$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that

$$g(t) = \lim_{n\to\infty} f(\delta_{\tau_n}(t)),$$

is well defined for each $t \in \mathbb{T}^*$ and

$$\lim_{n\to\infty} g(\delta_{\tau_n^{-1}}(t)) = \lim_{n\to\infty} g(\delta^{-1}_{\tau_n}(t)) = f(t)$$

for each $t \in \mathbb{T}^*$. Denote by $AA_\delta(\mathbb{T}, \mathbb{X})$ the set of all such functions.

(ii) A continuous function $f : \mathbb{T} \times \mathbb{X} \to \mathbb{X}$ is said to be almost automorphic if $f(t,x)$ is almost automorphic in $t \in \mathbb{T}^*$ uniformly for $x \in B$, where B is any bounded subset of $\mathbb{X}$. Denote by $AA_\delta(\mathbb{T} \times \mathbb{X}, \mathbb{X})$ the set of all such functions.

Definition 2.4. Let $\mathbb{T}$ be a periodic time scale under the matched space $(\mathbb{T}, \Pi, F, \delta)$.

(i) Let $f : \mathbb{T} \to \mathbb{X}$ be a bounded continuous function and the shift $\delta_\tau(t)$ is Δ-differentiable with rd-continuous bounded derivatives $\delta^{\Delta}_{\tau}(t) := \delta^{\Delta}(\tau, t)$ for all $t \in \mathbb{T}^*$. We say that f is Δ-almost automorphic if from every sequence $\{s_n\} \subset \widetilde{\Pi}$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that

$$g(t) = \lim_{n\to\infty} f(\delta_{\tau_n}(t))\delta^{\Delta}_{\tau_n}(t),$$

is well defined for each $t \in \mathbb{T}^*$ and

$$\lim_{n\to\infty} g(\delta_{\tau_n^{-1}}(t))\delta^{\Delta}_{\tau_n^{-1}}(t) = f(t)$$

for each $t \in \mathbb{T}^*$. Denote by $AA_\delta(\mathbb{T}, \mathbb{X})$ the set of all such functions.

(ii) A continuous function $f : \mathbb{T} \times \mathbb{X} \to \mathbb{X}$ is said to be Δ-almost automorphic if $f(t,x)$ is Δ-almost automorphic in $t \in \mathbb{T}^*$ uniformly for $x \in B$, where B is any bounded subset of $\mathbb{X}$. Denote by $AA_\delta(\mathbb{T} \times \mathbb{X}, \mathbb{X})$ the set of all such functions.

Note the above concepts under oriented-direction matched space can be found in [7].

3 Delay dynamic equations and models

In [5], the authors proposed some types of delay dynamic equations on time scales. The range of the delay functions should be a subset of the shift invariant number set, for example, when a time scale is a bi-direction shift invariant time scale (i.e., a periodic time scale in the sense of Definition 1.5), the shift invariant number set is the set formed by all the periods of $\mathbb{T}$. In [10], an n_0-order Δ-almost periodic theory of dynamic equations was established, and we considered the following almost periodic dynamic equation with variable delays under the matched space $(\mathbb{T}, F, \Pi, \delta)$:

$$x^{\Delta}(t) = S_A^{n_0}(t)x(t) + \sum_{i=1}^{n} S_f^{n_0}\big(t, x(\delta(\tau_i(t), t))\big),$$

where $A(t)$ is an $\Delta_{n_0}^{\delta}$-almost periodic matrix function on $\mathbb{T}$, $\tau_i(t) : \mathbb{T}^* \to \Pi^*$ is $\Delta_{n_0}^{\delta}$-almost periodic on $\mathbb{T}$ for every $i = 1, 2, \dots, n$, and $f \in C(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$ is $\Delta_{n_0}^{\delta}$-almost periodic uniformly in t for $x \in \mathbb{R}^n$.

In [8], the following almost periodic impulsive stochastic Lasota–Wazewska timescale model was considered.

$$\begin{cases} \Delta\big(x_i(t) + c_i(t)x_i(\delta_-(\tau_i, t))\big) = \Big[-\alpha_i(t)x_i(t) + \displaystyle\sum_{j=1}^{m} \beta_{ij}(t)e^{-\gamma_{ij}(t)x_j(\delta_-(\tau_{ij}, t))}\Big]\Delta t \\ \qquad + \displaystyle\sum_{j=1}^{m} H_{ij}\big(t, x_j(\delta_-(\sigma_{ij}, t))\big)\Delta\omega_j(t), & t \neq t_k, \\ \widetilde{\Delta} x_i(t_k) = x_i(t_k^+) - x_i(t_k^+) = I_{ik}(x_i(t_k)) + \alpha_{ik}x_i(t_k) + \nu_{ik}, & t = t_k; \end{cases}$$

see [8] for the biological background of the above model. Therefore, the shift invariance of time scales contributes to constructing the delay functions effectively in delay dynamic equations on time scales.

4 Almost translation invariance of time scales

In the literature [9], we employ an approximation method to introduce the concept of almost translation invariance of time scales (i.e., almost-complete closedness time scale).

Definition 4.1. We say $\mathbb{T}$ is an almost-complete closedness time scale (ACCTS) if for any given $\varepsilon_1 > 0$, there exist a constant $l(\varepsilon_1) > 0$ such that each interval of length $l(\varepsilon_1)$ contains a $\tau(\varepsilon_1)$ and sets $A_\tau^{\varepsilon_1}$ such that

$$d(\overline{\mathbb{T} \setminus A_\tau^{\varepsilon_1}}, \mathbb{T}^\tau) < \varepsilon_1$$

i.e., for any $\varepsilon_1 > 0$, the following set

$$\mathrm{E}\{\mathbb{T}, \varepsilon_1\} = \big\{\tau \in \Pi :\ d(\overline{\mathbb{T} \setminus A_\tau^{\varepsilon_1}}, \mathbb{T}^\tau) < \varepsilon_1\big\} := \Pi_{\varepsilon_1}$$

is relatively dense in Π. Here, τ is called the ε_1-translation number of $\mathbb{T}$, $l(\varepsilon_1)$ is called the inclusion length of $\mathrm{E}\{\mathbb{T}, \varepsilon_1\}$, and $\mathrm{E}\{\mathbb{T}, \varepsilon_1\}$ the ε_1-translation set of $\mathbb{T}$, $A_\tau^{\varepsilon_1}$ is called the ε_1-improper set of $\mathbb{T}$,

$$\mathscr{R}_{\mathbb{T}}(\tau, \varepsilon_1) := \mathbb{T} \cap \Big(\bigcup_{\tau \in \Pi_{\varepsilon_1}} \overline{\mathbb{T}^{-\tau} \setminus A_{-\tau}^{\varepsilon_1}} \Big)$$

the ε_1-main region of $\mathbb{T}$, where

$$A_{-\tau}^{\varepsilon_1} = (A_\tau^{\varepsilon_1})^{-\tau} := \big\{a - \tau :\ a \in A_\tau^{\varepsilon_1}\big\}.$$

Note that Definition 4.1 will include the concept of almost periodic time scales proposed in [3], which was developed in [5, 6]. This concept was applied to study double-almost periodic functions and solutions with double almost periodicity to dynamic equations on time scales (see [9]).

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