Analytic Representation and Some Properties of "Bulky" Links, Generated by Generalised Möbius–Listing's Bodies

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Natural forms affect all of us, not only for their beauty, but also for their diversity. It is still not known whether forms define the essence of the phenomena associated with them, or vice versa that is, forms are natural consequences of the phenomena. The essence of one "unexpected" phenomenon is as follows: Usually after one "full cutting", an object is split into two parts. The Möbius strip is a well-known exception, however, which still remains whole after cutting. In 2008–2009 author discovered a class of surfaces, which have following properties – after full cutting more than two surfaces appear, but this is a result for specific class of pure mathematical surfaces [2]. It turns out that three-dimensional Möbius Listing bodies, GML_m^n, (m is number of symmetry of the radial cross section and n is a number of twisting) which is a wide subclass of the Generalized Twisting and Rotated figures – shortly GTR_m^n – which, through their analytic representation, could yield more than two objects after only single cutting ([3] or [5]). These are not only theoretical results, as can be proved by real-life examples. Many classical objects (torus with different forms of radial cross sections, helicoid, helix, Möbius strip, … etc.) are elements of this wide class of GTR_m^n figures, so it is important to study the similarity and difference between these figures and surfaces. One possible application of these results is assumed in the description of the properties of the middle surfaces in the theory of elastic shells [6].

Based on the one form of analytical representation

$$\begin{cases} X(\tau,\theta)=\Big(R+r(\tau,\theta)\cos\Big(\psi+\dfrac{n\theta}{m}\Big)\Big)\cos(\theta),\\[2mm] Y(\tau,\theta)=\Big(R+r(\tau,\theta)\cos\Big(\psi+\dfrac{n\theta}{m}\Big)\Big)\sin(\theta),\\[2mm] Z(\tau,\theta)=r(\tau,\theta)\sin\Big(\psi+\dfrac{n\theta}{m}\Big), \end{cases} \tag{1}$$

and on the definition of operation of cutting defined earlier [2, 4], some basic questions to be answered appear, for example:

1. How many objects appear after cutting of the GML_m^n surfaces or bodies?
2. What type of GML_m^n surfaces or bodies appear after cutting (this question for Möbius strip was formulated for the first time by Sosinski see e.g. [1])?
3. What is a link-structure of the surfaces or bodies, which appear after cutting?
4. What are shapes of radial cross sections of the geometric objects which appear after cutting of GML_m^n surfaces or bodies?
5. How many different combinations of geometric objects (in the sense of shapes of the radial cross sections) appear after cutting for arbitrary number m in GML_m^n?

6. What are differential geometric characteristics of GML_m^n surfaces or bodies?

At this stage, we unfortunately do not have answers to all of these questions raised in the case of arbitrary values of m, but some particular cases were reported by the author and his colleagues [2, 3, 5].

In this report we give some general results.

Remark.

- **A.** If m is an even number, then for different n (more precisely, if $\gcd(m,n)=1$) – after one full cutting of GML_m^n bodies, maximum $m/2+1$ independent geometric objects appear (this number depends also on the geometric place of the cutting line in the cross section of body), i.e. link-$(m/2+1)$ appear and only one element has structure similar to figure before cutting;

- **B.** If m is an odd number, then for different n (more precisely, if $\gcd(m,n)=1$) – after one full cutting of GML_m^n bodies, maximum $[m/2]+2$ independent geometric objects appear (in the cross section of body), i.e. link-$([m/2]+2)$ appear and only one element has structure similar to figure before cutting; $[m/2]$ is the integer part of number $m/2$; (see example for $m=5$ and $n=1$ in Figure 1).

- **C.** If m is even number, then always exist some values of n, such that after one full cutting of GML_m^n bodies only 1 independent geometric object appears (for this, the cutting line should include the center of symmetry of the radial cross section of the body), i.e. knot (link-1 appears, whose index is defined by $\gcd(m,n)=1$;

  ⋆ If m is an even number, then always some values of n exist, and the phenomenon of the Möbius strip is realized (see example for $m=6$ and $n=1$ in Figure 2)!

- **D.** If m is odd number, then there exist some values of n, such that after one cutting of GML_m^n at least 2 independent geometric objects appear (for this the cutting line should include the center of symmetry of the radial cross section of body), i.e. (link-2 appears), whose index is defined by $\gcd(m,n)=1$;

  ⋆ If m is an odd number, then for any value of the parameter n, the phenomenon of the Möbius band is never realized!

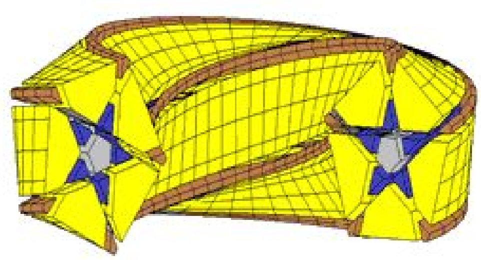

Figure 1.

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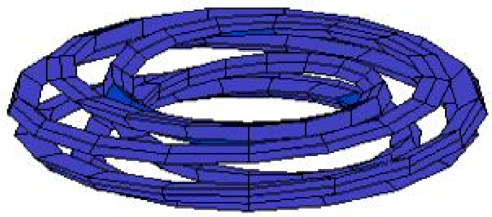

Figure 2.

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