Existence of Rapidly Varying Solutions of Second Order Half-Linear Differential Equations

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1 Introduction

This paper is concerned with positive solutions of generalized Thomas–Fermi equations of the form

$$(p(t)\varphi_\alpha(x'))' = q(t)\varphi_\alpha(x), \ \ t \geqq a \ \ \big(\varphi_\gamma(\xi) = |\xi|^{\gamma-1}\xi = |\xi|^\gamma \operatorname{sgn}\xi, \ \ \xi \in \mathbb{R}, \ \ \gamma > 0\big) \tag{A}$$

for which the following conditions are always assumed to hold:

(a) α is a positive constant;

(b) $p, q : [a,\infty) \to (0,\infty)$, $a \geqq 0$ are continuous functions;

(c) $p(t)$ satisfies that either

$$\int\limits_a^\infty \frac{dt}{p(t)^{\frac{1}{\alpha}}} = \infty \quad \bigg(P(t) = \int\limits_a^t \frac{ds}{p(s)^{\frac{1}{\alpha}}}\bigg) \tag{1.1}$$

or

$$\int\limits_a^\infty \frac{dt}{p(t)^{\frac{1}{\alpha}}} < \infty \quad \bigg(\pi(t) = \int\limits_t^\infty \frac{ds}{p(s)^{\frac{1}{\alpha}}}\bigg). \tag{1.2}$$

By a positive solution on an interval J of the differential equation (A) we mean a function $x : J \to (0,\infty)$ which is continuously differentiable on J together with $p(t)\varphi_\alpha(x'(t))$ and satisfies (A) there.

Since the publication of the book [11] of Marić in the year 2000, the class of rapidly varying functions in the sense of Karamata [7] is a well-suited framework for the asymptotic analysis of nonoscillatory solutions of second order linear differential equation of the form

$$x''(t) = q(t)x(t), \ \ q(t) > 0. \tag{B}$$

2 Definitions of rapidly varying functions

Definition 2.1. A measurable function $f : [a,\infty) \to (0,\infty)$ is said to be a rapidly varying of index ∞ if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \begin{cases} \infty & \text{for } \ \lambda > 1, \\ 0 & \text{for } \ 0 < \lambda < 1. \end{cases}$$

Moreover, a measurable function $f : [a,\infty) \to (0,\infty)$ is said to be a rapidly varying of index $-\infty$ if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \begin{cases} 0 & \text{for } \ \lambda > 1, \\ \infty & \text{for } \ 0 < \lambda < 1. \end{cases}$$

The totality of rapidly varying functions of index ∞ (or $-\infty$) is denoted by $\mathrm{RPV}(\infty)$ (or $\mathrm{RPV}(-\infty)$). The functions

$$f(t) = e^{kt}, \quad f(t) = \exp\{t^k\}, \quad f(t) = \exp\{e^t\},$$

and

$$f(t) = e^{-kt}, \quad f(t) = \exp\{-t^k\}, \quad f(t) = \exp\{-e^t\}$$

belong to the $\mathrm{RPV}(\infty)$ and $\mathrm{RPV}(-\infty)$, respectively, for all $k > 0$. For the rapidly varying functions the reader referred to the book of Bingham, Goldie and Teugels [1].

3 Definitions of generalized rapidly varying functions

In 2004, Jaroš and Kusano [6] set up the framework of positive solutions which is suitable for the asymptotic analysis of the self-adjoint differential equation

$$(p(t)x'(t))' = q(t)x(t), \tag{C}$$

because the asymptotic behavior of solutions of (C) depends heavily on the function $P(t)$ or $\pi(t)$ given by (1.1) or (1.2), respectively. Therefore, they needed to make the properly generalizing the class of rapidly varying functions in the sense of Karamata. In the generalization use is made of a positive function $R(t)$ which is continuously differentiable on $[a,\infty)$ and satisfies

$$R'(t) > 0 \ \text{ for } \ t \geq t_0 \ \text{ and } \ \lim_{t\to\infty} R(t) = \infty.$$

The inverse function of $R(t)$ is denoted by $R^{-1}(t)$.

Definition 3.1. A measurable function $f : [a,\infty) \to (0,\infty)$ is said to be rapidly varying with index ∞ (or $-\infty$) with respect to $R(t)$ if the function $f \circ R^{-1}(t)$ is rapidly varying with index ∞ (or $-\infty$) in the sense of Definition 2.1. The set of all rapidly varying functions with index ∞ (or $-\infty$) with respect to $R(t)$ is denoted by $\mathrm{RPV}_R(\infty)$ (or $RPV_R(-\infty)$), that is, $f \in \mathrm{RPV}_R(\infty)$ satisfies

$$\lim_{t\to\infty} \frac{f(R^{-1}(\lambda t))}{f(R^{-1}(t))} = \begin{cases} \infty & \text{for } \ \lambda > 1, \\ 0 & \text{for } \ 0 < \lambda < 1, \end{cases}$$

and that $f \in \mathrm{RPV}_R(-\infty)$ satisfies

$$\lim_{t\to\infty} \frac{f(R^{-1}(\lambda t))}{f(R^{-1}(t))} = \begin{cases} 0 & \text{for } \ \lambda > 1, \\ \infty & \text{for } \ 0 < \lambda < 1. \end{cases}$$

4 Main result

The equation (A_0) $(\varphi_\alpha(x'))' + q(t)\varphi_\alpha(x) = 0$ has been the object of intensive investigations from the late 20th century because of the fact that it has many fundamental qualitative properties in common with those of the linear equation

$$x''(t) + q(t)x(t) = 0. \tag{D}$$

This fact was first discovered by Elbert [4] and Mirzov [12], who showed in particular that the classical Sturmian comparison and separation theorems for (D) can be carried over to (A_0) almost verbatim and literatim. Since the pioneering work of Elbert and Mirzov much efforts have been directed towards an in-depth analysis of the similarity existing between (A_0) and (D). The reader is referred to the papers [2,3,5,8–10] for typical results on oscillation and/or nonoscillation of (A_0) which are derived in this way on the basis of the well-developed linear oscillation theory of (D).

Theorem A (V. Marić [11]). *The equation* (B) *has a fundamental set of solutions* $\{x_1(t), x_2(t)\}$ *such that*

$$x_1 \in \mathrm{RPV}(-\infty) \ \textit{ and } \ x_2 \in \mathrm{RPV}(\infty)$$

if and only if

$$\lim_{t\to\infty} t \int\limits_t^{\lambda t} q(s)\,ds = \infty \ \textit{ for all } \ \lambda > 1.$$

The purpose of this paper is to obtain the conditions of the existence of rapidly varying solutions of (A) based on the above result of Marić. It will turn out that $R(t) = P(t)$ or $R(t) = 1/\pi(t)$ is the best choice of $R(t)$ for the analysis of the equation (A) with $p(t)$ subject to (1.1) or (1.2), respectively.

We will establish the conditions for the existence of rapdily varying solutions of the equation (A) with the case where the function $p(t)$ satisfies the condition (1.1) or (1.2), respectively.

Theorem 4.1. *Suppose that the function $p(t)$ satisfies the condition* (1.1) *and*

$$\lim_{t\to\infty} P(t)^{\alpha} \int\limits_t^{P^{-1}(\lambda P)(t)} q(s)\,ds = \infty$$

are satisfied for all $\lambda > 1$. Then, the equation (A) *possesses*

$$x_1 \in \mathrm{RPV}_P(-\infty) \ \textit{ and } \ x_2 \in \mathrm{RPV}_P(\infty).$$

Theorem 4.2. *Suppose that the function $p(t)$ satisfies the condition* (1.2) *and*

$$\lim_{t\to\infty} \frac{1}{\pi(t)} \int\limits_t^{(\frac{1}{\pi})^{-1}(\frac{\lambda}{\pi})(t)} \pi(s)^{\alpha+1} q(s)\,ds = \infty$$

are satisfied for all $\lambda > 1$. Then, the equation (A) *possesses*

$$x_1 \in \mathrm{RPV}_{\frac{1}{\pi}}(-\infty) \ \textit{ and } \ x_2 \in \mathrm{RPV}_{\frac{1}{\pi}}(\infty).$$

5 Examples

We here present two examples illustrating the results developed in the preceding Theorems 4.1 and 4.2.

Example 5.1. Consider the differential equation

$$(e^{-\alpha t}\varphi_{\alpha}(x'))' = \alpha e^{t}\varphi_{\alpha}(x), \ \ t \geqq 0, \tag{5.1}$$

where α is as in (A). The function $P(t)$ defined by (1) can be taken to be $P(t) = e^t$. Since, for all $\lambda > 1$,

$$\lim_{t\to\infty} P(t)^{\alpha} \int\limits_t^{P^{-1}(\lambda P)(t)} q(s)\,ds = \alpha(\lambda-1)e^{(\alpha+1)t} \longrightarrow \infty \ \text{ as } \ t \to \infty,$$

we see from Theorem 4.1 that the equation (5.1) possesses rapidly varying solutions such that

$$x_1 \in \mathrm{RPV}_{e^t}(-\infty) \ \text{ and } \ x_2 \in \mathrm{RPV}_{e^t}(\infty).$$

It is easy to check that $x(t) = \exp\{-e^t\} \in \mathrm{RPV}_{e^t}(-\infty)$ is one such solution of (5.1).

Example 5.2. Consider the differential equation

$$\big(t^\alpha (\log t)^{2\alpha} \varphi_\alpha(x')\big)' = q(t)\varphi_\alpha(x), \ \ t \geqq e, \ \ q(t) \sim \frac{\alpha(\log t)^{2\alpha}}{t} \ \text{ as } \ t \to \infty, \tag{5.2}$$

where α is as in (A) and the symbol $\sim$ is used to denote the asymptotic equivalence

$$f(t) \sim g(t) \ \text{ as } \ t \to \infty \iff \lim_{t\to\infty} \frac{f(t)}{g(t)} = 1.$$

Since $\pi(t) \sim (\log t)^{-1}$, $t \to \infty$, we have, for all $\lambda > 1$,

$$\frac{1}{\pi(t)} \int\limits_t^{(\frac{1}{\pi})^{-1}(\frac{\lambda}{\pi})(t)} \pi(s)^{\alpha+1} q(s)\, ds \sim \alpha\lambda \log t \longrightarrow \infty \ \text{ as } \ t \to \infty,$$

from which it follows from Theorem 4.2 that the equation (5.2) possesses rapidly varying solutions such that

$$x_1 \in \mathrm{RPV}_{\log t}(-\infty) \ \text{ and } \ x_2 \in \mathrm{RPV}_{\log t}(\infty).$$

$x(t) = t \in \mathrm{RPV}_{\log t}(\infty)$ is one such solution of (5.2).

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