On a Delay Parameter Optimization Problem: Existence, Sensitivity of a Functional Minimum, Necessary Optimality Conditions

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Let $t_1 > t_0$ and $\theta_2 > \theta_1 > 0$ be given numbers with $t_0 + \theta_2 < t_1$; let $O \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^r$ be open and compact sets. Let us consider the function $F(t,x,y,u) = (f^0, f)$, $(t,x,y,u) \in I \times O^2 \times U$ satisfying the standard conditions: for almost all fixed $t \in I = [t_0, t_1]$ the function $F(t,\,\cdot\,) : O^2 \times U \to \mathbb{R}^{1+n}$ is continuous in $(x,y,u) \in O^2 \times U$ and continuously differentiable in $(x,y) \in O^2$; for each fixed $(x,y,u) \in O^2 \times U$ the functions $F(t,x,y,u)$, $F_x(t,\,\cdot\,)$ and $F_y(t,\,\cdot\,)$ are measurable on I; for any compact set $K \subset O$ there exists a function $m_K(t) \in L_1(I, \mathbb{R}_+)$, $\mathbb{R}_+ = [0,\infty)$ such that

$$|F(t,x,y,u)| + |F_x(t,\,\cdot\,)| + |F_y(t,\,\cdot\,)| \le m_K(t)$$

for all $(x,y,u) \in K^2 \times U$ and for almost all $t \in I$.

Furthermore, by Ω we denote the set of measurable control functions $u : I \to U$; the initial function $\varphi_0(t) \in C(I_1, O)$, where $I_1 = [\widehat{\tau}, t_1]$, $\widehat{\tau} = t_0 - \theta_2$; the initial vector $x_{00} \in O$, the function $q^0(\tau, x) \in C(I_2 \times O, \mathbb{R})$, where $I_2 = [\theta_1, \theta_2]$.

To each element $w = (\tau, u(t)) \in W = I_2 \times \Omega$ we set in correspondence the controlled delay functional differential equation

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau), u(t)\big) \tag{1}$$

with the initial condition

$$x(t) = \varphi_0(t), \ \ t \in [\widehat{\tau}, t_0), \quad x(t_0) = x_{00}. \tag{2}$$

Definition 1. Let $w = (\tau, u(t)) \in W$. A function $x(t) = x(t;w) \in O$, $t \in [\widehat{\tau}, t_1]$, is called a solution of the equation (1) with the initial condition (2) or a solution corresponding to the element w and defined on the interval $[\widehat{\tau}, t_1]$ if it satisfies the condition (2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies the equation (1) almost everywhere on $[t_0, t_1]$.

By W_0 we denote the set of all $w \in W$ elements for which there exist solutions $x(t;w)$ defined on the interval I. In the sequel it is assumed that $W_0 \neq \varnothing$.

Definition 2. An element $w_0 = (\tau_0, u_0(t)) \in W_0$ is said to be optimal or a solution of the problem (1)–(3) if for an arbitrary element $w \in W_0$ the inequality

$$\begin{aligned} J(w_0) &= q^0(\tau_0, x_0(t_1)) + \int_{t_0}^{t_1} f^0\big(t, x_0(t), x_0(t-\tau_0), u_0(t)\big)\,dt \\ &\le J(w) = q^0(\tau, x(t_1)) + \int_{t_0}^{t_1} f^0\big(t, x(t), x(t-\tau), u(t)\big)\,dt \end{aligned} \tag{3}$$

holds, where $x_0(t) = x(t;w_0)$, $x(t) = x(t;w)$.

The problem (1)–(3) is called the optimization problem of delay parameter.

Theorem 1. *There exists an optimal element w_0 if the following conditions hold:*

1.1 *there exists a compact set $K_0 \subset O$ such that for an arbitrary $w \in W_0$*

$$x(t; w) \in K_0, \ \ t \in I;$$

1.2 *the set*

$$P_F(t, x, y) = \big\{(p^0, p)^T \in \mathbb{R}^{1+n} : \ \exists\, u \in U, \ p^0 \geq f^0(t, x, y, u), \ p = f(t, x, y, u)\big\}$$

is convex for all fixed $(t, x, y) \in I \times K_0^2$.

Remark 1. Let U be the convex set. Let $f(t, x, y, u) = A(t, x, y) + B(t, x, y)u$ and let the function $f^0(t, x, y, u)$ be convex in $u \in U$, then the condition 1.2 of the Theorem 1 holds.

Theorem 2. *Let the conditions* 1.1 *and* 1.2 *of Theorem* 1 *hold. Then for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every $(x_{0\delta}, \varphi_\delta, q_\delta, G_\delta)$, where $x_{0\delta} \in O$, $\varphi_\delta \in C(I_1, O)$, $q_\delta^0 \in C(I_2 \times O, \mathbb{R})$, $G_\delta = (g_\delta^0, g_\delta)$ satisfying the condition*

$$|x_{00} - x_{0\delta}| + \|\varphi_0 - \varphi_\delta\| + \|q^0 - q_\delta^0\| + \|G_\delta\|_{K_1} \leq \delta,$$

there exists a solution $w_\delta = (\tau_\delta, u_\delta(t))$ of the perturbed optimal problem

$$\begin{gathered}
\dot{x}(t) = f\big(t, x(t), x(t-\tau), u(t)\big) + g_\delta\big(t, x(t), x(t-\tau)\big), \\
x(t) = \varphi_\delta(t), \ \ t \in [\widehat{\tau}, t_0), \quad x(t_0) = x_{0\delta}, \\
J(w; \delta) = q_\delta^0(\tau, x(t_1)) + \int\limits_{t_0}^{t_1} \Big[f^0\big(t, x(t), x(t-\tau), u(t)\big) + g_\delta^0\big(t, x(t), x(t-\tau)\big)\Big]\, dt \longrightarrow \min
\end{gathered}$$

and $|J(w_0) - J(w_\delta; \delta)| < \varepsilon$. Here, the functions $G_\delta(t, x, y)$ satisfy the standard conditions on the set $I \times O^2$ and

$$\int\limits_I \sup\Big\{|G_\delta(t, x, y)| + |G_{\delta x}(t, \cdot\,)| + |G_{\delta y}(t, \cdot\,)| : \ (x, y) \in K_1^2\Big\}\, dt \leq const,$$

where $K_1 \subset O$ is a compact set containing a neighborhood of K_0;

$$\begin{gathered}
G_{\delta x} = \frac{\partial}{\partial x}\, G_\delta, \quad \|\varphi_0 - \varphi_\delta\| = \sup\big\{|\varphi_0(t) - \varphi_\delta(t)| : \ t \in I_1\big\}, \\
\|q_0 - q_\delta\| = \sup\Big\{|q_0(\tau, x) - q_\delta(\tau, x)| : \ (\tau, x) \in I_2 \times K_1\Big\}, \\
\|G_\delta\|_{K_1} = \sup\bigg\{\bigg|\int\limits_{s_1}^{s_2} G_\delta(t, x, y)\, dt\bigg| : \ (s_1, s_2, x, y) \in I^2 \times K_1^2\bigg\}.
\end{gathered}$$

Theorem 3. *Let the conditions* 1.1 *and* 1.2 *of Theorem* 1 *hold. Then for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every $(x_{0\delta}, \varphi_\delta, q_\delta^0, G_\delta)$, where $x_{0\delta} \in O$, $\varphi_\delta \in C(I_1, O)$, $q_\delta^0 \in C(I_2 \times O, \mathbb{R})$, $G_\delta(t, x, y, u) = (g_\delta^0, g_\delta)$ satisfying the conditions*

$$|x_{00} - x_{0\delta}| + \|\varphi_0 - \varphi_\delta\| + \|q^0 - q_\delta^0\| + \|G_\delta\|_1 \leq \delta$$

and the set $P_{F+G_\delta}(t,x,y)$ is convex, there exists a solution $w_\delta=(\tau_\delta,u_\delta(t))$ of the perturbed optimal problem

$$\begin{gathered}\dot x(t)=f\big(t,x(t),x(t-\tau),u(t)\big)+g_\delta\big(t,x(t),x(t-\tau),u(t)\big),\\ x(t)=\varphi_\delta(t),\ \ t\in[\widehat{\tau},t_0),\quad x(t_0)=x_{0\delta},\\ J(w;\delta)=q_\delta^0(\tau,x(t_1))+\int\limits_{t_0}^{t_1}\Big[f^0\big(t,x(t),x(t-\tau),u(t)\big)+g_\delta^0\big(t,x(t),x(t-\tau),u(t)\big)\Big]\,dt\longrightarrow\min\end{gathered}$$

and $|J(w_0)-J(w_\delta;\delta)|<\varepsilon$. Here, the functions $G_\delta(t,x,y,u)$ satisfy the standard conditions on the set $I\times O^2\times U$ and

$$\int\limits_I\sup\Big\{|G_{\delta x}(t,x,y,u)|+|G_{\delta y}(t,\,\cdot\,)|:\ (x,y,u)\in K_1^2\times U\Big\}\,dt\le const;$$

$$\|G_\delta\|_1=\int\limits_I\sup\big\{|G_\delta(t,x,y,u)|:\ (x,y,u)\in K_1^2\times U\big\}\,dt.$$

Theorem 4. *Let $w_0=(\tau_0,u_0(t)),\tau_0\in(\theta_1,\theta_2)$, be an optimal element and the following conditions hold:*

4.1 *the initial function $\varphi_0(t)$ is absolutely continuous and $\dot\varphi_0(t)$ is bounded;*

4.2 *the function $F(t,x,y,u_0(t)),(t,x,y)\in I\times O^2$ is bounded;*

4.3 *there exist the finite limit*

$$\lim_{(v_1,v_2)\to(v_{10},v_{20})}\big[F(v_1,u_0(t))-F(v_2,u_0(t))\big]=F_0=(f_0^0,f_0)^T,$$

where $v_1,v_2\in I\times O^2$,

$$v_{10}=\big(t_0+\tau_0,x_0(t_0+\tau_0),x_0\big),\quad v_{20}=\big(t_0+\tau_0,x_0(t_0+\tau_0),\varphi_0(t_0)\big).$$

Then the following conditions hold:

4.4 *the condition for the optimal delay parameter τ_0*

$$\begin{aligned}-\frac{\partial}{\partial\tau}\,q_0(\tau_0,x_0(t_1))&=-f_0^0+\psi(t_0+\tau_0)f_0\\ &\quad+\int\limits_{t_0}^{t_0+\tau_0}\big\{-f_y^0[t+\tau_0]+\psi(t)f_y[t+\tau_0]\big\}\dot\varphi_0(t-\tau_0)\,dt\\ &\quad+\int\limits_{t_0+\tau_0}^{t_1}\big\{-f_y^0[t+\tau_0]+\psi(t)f_y[t+\tau_0]\big\}\dot x_0(t-\tau_0)\,dt;\end{aligned}$$

4.5 *the condition for the optimal control $u_0(t)$*

$$\begin{aligned}&-f^0\big(t,x_0(t),x_0(t-\tau_0),u_0(t)\big)+\psi(t)f\big(t,x_0(t),x_0(t-\tau_0),u_0(t)\big)\\ &\qquad=\max_{u\in U}\Big[-f^0(t,x_0(t),x_0(t-\tau_0),u)+\psi(t)f(t,x_0(t),x_0(t-\tau_0),u)\Big].\end{aligned}$$

Here $\psi(t)$ is the solution of the equation

$$\psi(t) = f_x^0[t] - \psi(t) f_x[t] + \chi(t+\tau_0)\big\{ f_y^0[t+\tau_0] - \psi(t+\tau_0) f_y[t+\tau_0] \big\}, \quad t \in [t_0, t_1]$$

with the initial condition

$$\psi(t_1) = -q_x^0(\tau_0, x_0(t_1));$$
$$f_x^0[t] = f_x^0\big(t, x_0(t), x_0(t-\tau_0), u_0(t)\big).$$

Some comments. The theorems of existence and sensitivity of the functional minimum for optimal problems involving various functional differential equations with fixed delay are given in [1–3]. Theorems 1–3 and Theorem 4 are proved by the scheme given in [3] and [4], respectively.

References

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