Variation Formulas of Solutions for Nonlinear Controlled Functional Differential Equations with Constant Delay and the Discontinuous Initial Condition

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Let $O \subset \mathbb{R}^n$ and $U_0 \subset \mathbb{R}^r$ be open sets. Let $\theta_2 > \theta_1 > 0$ be given numbers and n-dimensional function $f(t,x,y,u)$ satisfy the following conditions: for almost all fixed $t \in I = [a,b]$ the function $f(t,\,\cdot\,) : O^2 \times U_0 \to \mathbb{R}^n$ is continuously differentiable; for each fixed $(x,y,u) \in O^2 \times U_0$ the functions $f(t,x,y,u)$, $f_x(t,\,\cdot\,)$, $f_y(t,\,\cdot\,)$ and $f_u(t,\,\cdot\,)$ are measurable on I; for compact sets $K \subset O$ and $U \subset U_0$ there exists a function $m_{K,U}(t) \in L_1(I,[0,\infty))$ such that

$$|f(t,x,y,u)| + |f_x(t,\,\cdot\,)| + |f_y(t,\,\cdot\,) + |f_u(t,\,\cdot\,)| \le m_{K,U}(t)$$

for all $(x,y,u) \in K^2 \times U$ and for almost all $t \in I$. Furthermore, Φ is the set of continuous initial functions $\varphi : I_1 = [\widehat{\tau}, b] \to O$, $\widehat{\tau} = a - \theta_2$ and Ω is the set of measurable control functions $u : I \to U$ with $\operatorname{cl} u(I)$ is a compact set and $\operatorname{cl} u(I) \subset U$.

To each element $\mu = (t_0, \tau, x_0, \varphi(t), u(t)) \in \Lambda = [a,b) \times [\theta_1, \theta_2] \times O \times \Phi \times \Omega$ we assign the delay controlled functional differential equation

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau), u(t)\big) \tag{1}$$

with the discontinuous initial condition

$$x(t) = \varphi(t), \ \ t \in [\widehat{\tau}, t_0), \quad x(t_0) = x_0. \tag{2}$$

Condition (2) is said to be the discontinuous initial condition because, in general, $x(t_0) \neq \varphi(t_0)$.

Definition. Let $\mu = (t_0, \tau, x_0, \varphi(t), u(t)) \in \Lambda$. A function $x(t) = x(t;\mu) \in O$, $t \in [\widehat{\tau}, t_1]$, $t_1 \in (t_0, b]$ is called a solution of equation (1) with the initial condition (2) or the solution corresponding to μ and defined on the interval $[\widehat{\tau}, t_1]$ if it satisfies condition (2) and is absolutely continuous on the interval $[t_0, t_1]$ and satisfies equation (1) almost everywhere on $[t_0, t_1]$.

Let us introduce the set of variation:

$$V = \Big\{ \delta\mu = (\delta t_0, \delta\tau, \delta x_0, \delta\varphi, \delta u) : \ \delta t_0| \le \alpha, \ |\delta\tau| \le \alpha, \ |\delta x_0| \le \alpha, \\ \delta\varphi = \sum_{i=1}^{k} \lambda_i \delta\varphi_i, \ \delta u = \sum_{i=1}^{k} \lambda_i \delta u_i, \ |\lambda_i| \le \alpha, \ i = \overline{1,k} \Big\},$$

where $\delta\varphi_i \in \Phi - \varphi_0$, $\delta u_i \in \Omega - u_0$, $i = \overline{1,k}$. Here $\varphi_0 \in \Phi, u_0 \in \Omega$ are fixed functions and $\alpha > 0$ is a fixed number.

Let $\mu_0 = (t_{00}, \tau_0, x_{00}, \varphi_0(t), u_0(t)) \in \Lambda$ be a fixed element, where $t_{00}, t_{10} \in (a,b)$, $t_{00} < t_{10}$ and $\tau_0 \in (\theta_1, \theta_2)$. Let $x_0(t)$ be the solution corresponding to μ_0. There exist numbers $\delta_1 > 0$ and $\varepsilon_1 > 0$

such that for arbitrary $(\varepsilon,\delta\mu)\in(0,\varepsilon_1)\times V$, we have $\mu_0+\varepsilon\delta\mu\in\Lambda$, and the solution $x(t;\mu_0+\varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau},t_{10}+\delta_1]\subset I_1$ corresponds to it (see, [2, Theorem 1.4]).

By the uniqueness, the solution $x(t;\mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau},t_{10}+\delta_1]$. Therefore, we can assume that the solution $x_0(t)$ is defined on the whole interval $[\widehat{\tau},t_{10}+\delta_1]$. Now we introduce the increment of the solution $x_0(t)=x(t;\mu_0)$:

$$\Delta x(t;\varepsilon\delta\mu)=x(t;\mu_0+\varepsilon\delta\mu)-x_0(t),\ \ (t,\varepsilon,\delta\mu)\in[\widehat{\tau},t_{10}+\delta_1]\times(0,\varepsilon_1)\times V.$$

Theorem 1. *Let the following conditions hold:*

1) $t_{00}+\tau_0<t_{10}$;
2) *the function* $\varphi_0(t)$ *is absolutely continuous and* $\dot\varphi_0(t),t\in I_1$ *is bounded;*
3) *the function* $f(w,u)$, *where* $w=(t,x,y)\in I\times O^2$ *is bounded on* $I\times O^2\times U_0$;
4) *there exists the finite limit*
$$\lim_{w\to w_0} f(w,u_0(t))=f^-,\ \ w\in(a,t_{00}]\times O^2,$$
where $w_0=(t_{00},\varphi_0(t_{00}),\varphi_0(t_{00}-\tau_0))$.
5) *there exist the finite limits*
$$\lim_{(w_1,w_2)\to(w_1^0,w_2^0)}\big[f(w_1,u_0(t))-f(w_2,u_0(t))\big]=f_1,\ \ w_1,w_2\in(a,b)\times O^2,$$
where
$$w_1^0=(t_{00},x_0(t_{00}+\tau_0),x_{00}),\ \ w_2^0=(t_{00},x_0(t_{00}+\tau_0),\varphi_0(t_{00})).$$

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ *and* $\delta_2\in(0,\delta_1)$ *with* $t_{10}-\delta_2>t_{00}+\tau_0$ *such that for arbitrary* $(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times(0,\varepsilon_2)\times V^-$, *where* $V^-=\{\delta\mu\in V:\delta t_0\le 0\}$, *we have*

$$\Delta x(t;\varepsilon\delta\mu)=\varepsilon\delta x(t;\delta\mu)+o(t;\varepsilon\delta\mu). \tag{3}$$

Here

$$\delta x(t;\delta\mu)=-Y(t_{00};t)f^-\delta t_0+\beta(t;\delta\mu), \tag{4}$$

$$\beta(t;\delta\mu)=Y(t_{00};t)\delta x_0-Y(t_{00}+\tau_0;t)f_1\delta t_0-\bigg[Y(t_{00}+\tau_0;t)f_1+\int_{t_{00}}^{t}Y(\xi;t)f_y[\xi]\dot x_0(\xi-\tau_0)\,d\xi\bigg]\delta\tau$$
$$+\int_{t_{00}-\tau_0}^{t}Y(\xi+\tau_0;t)f_y[\xi+\tau_0]\delta\varphi(\xi)\,d\xi+\int_{t_{00}}^{t}Y(\xi;t)f_u[\xi]\delta u(\xi)\,d\xi,$$

where it is assumed that

$$\int_{t_{00}}^{t}Y(\xi;t)f_y[\xi]\dot x_0(\xi-\tau_0)\,d\xi=\int_{t_{00}}^{t_{00}+\tau_0}Y(\xi;t)f_y[\xi]\dot\varphi_0(\xi-\tau_0)\,d\xi+\int_{t_{00}+\tau_0}^{t}Y(\xi;t)f_y[y]\dot x_0(\xi-\tau_0)\,d\xi.$$

Next, $Y(\xi;t)$ *is the* $n\times n$*-matrix function satisfying the equation*

$$Y_\xi(\xi;t)=-Y(\xi;t)f_x[\xi]-Y(\xi+\tau_0;t)f_y[\xi+\tau_0],\ \ \xi\in[t_{00},t]$$

and the condition

$$Y(\xi;t)=\begin{cases} H & \text{for } \xi=t,\\ \Theta & \text{for } \xi>t,\end{cases}$$

H is the identity matrix and Θ is the zero matrix; $f_y=\frac{\partial}{\partial y}f$, $f_y[\xi]=f_y(\xi,x_0(\xi),x_0(\xi-\tau_0),u_0(\xi))$.

Some comments. The expression (3) is called the variation formula of a solution.

The addend $-Y(t_{00};t)f^{-}\delta t_0-Y(t_{00}+\tau_0;t)f_1\delta t_0$ in the formula (4) is the effect of the discontinuous initial condition (2) and perturbation of the initial moment t_{00}.

The addend

$$-\bigg[Y(t_{00}+\tau_0;t)f_1+\int\limits_{t_{00}}^{t}Y(\xi;t)f_{x_1}[\xi]\dot{x}_0(\xi-\tau_0)\,d\xi\bigg]\delta\tau$$

in formula (4) is the effect of the discontinuous initial condition (2) and perturbation of the delay parameter τ_0.

The expression

$$Y(t_{00};t)\delta x_0+\int\limits_{t_{00}-\tau_0}^{t}Y(\xi+\tau_0;t)f_{x_1}[\xi+\tau_0]\delta\varphi(\xi)\,d\xi$$

in formula (4) is the effect of perturbations of the initial vector x_{00} and the initial function $\varphi_0(t)$.

The expression

$$\int\limits_{t_{00}}^{t}Y(\xi;t)f_u[\xi]\delta u(\xi)\,d\xi$$

is the effect of perturbation of the control function $u_0(t)$. Finally, we note that in [4] variation formulas of solutions were proved for equation (1) with the discontinuous initial condition (2) in the case when the initial moment and delay variations have the same signs. In the present paper variation formulas of solutions are obtained with respect to wide classes of variations (see, V^- and V^+). The variation formulas of solutions for various classes of controlled delay functional differential equations, without perturbations of delay, are proved in [1,3].

Theorem 2. *Let the conditions* 1)–3) *and* 5) *of Theorem* 1 *hold. Moreover, there exists the finite limit*

$$\lim_{w\to w_0}f(w,u_0(t))=f^{+},\ \ w\in[t_{00},b)\times O^2.$$

Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$, with $t_{10}-\delta_2>t_{00}+\tau_0$ such that for arbitrary $(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times(0,\varepsilon_2)\times V^{+}$, where $V^{+}=\{\delta\mu\in V:\delta t_0\geq 0\}$, formula (4) *holds, where*

$$\delta x(t;\delta\mu)=-Y(t_{00};t)f^{+}\delta t_0+\beta(t;\delta\mu).$$

Theorem 3. *Let the conditions* 1)–5) *of Theorem* 1 *and the condition* 6) *hold. Moreover, $f^{-}=f^{+}:=\widehat{f}$. Then there exist numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$, with $t_{10}-\delta_2>t_{00}+\tau_0$ such that for arbitrary $(t,\varepsilon,\delta\mu)\in[t_{10}-\delta_2,t_{10}+\delta_2]\times(0,\varepsilon_2)\times V$, formula* (4) *holds, where*

$$\delta x(t;\delta\mu)=-Y(t_{00};t)\widehat{f}\delta t_0+\beta(t;\delta\mu).$$

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