Investigation of Transcendental Boundary Value Problems Using Lagrange Interpolation

A. Rontó

Institute of Mathematics, Academy of Sciences of Czech Republic, Brno, Czech Republic
E-mail: ronto@math.cas.cz

M. Rontó

Institute of Mathematics, University of Miskolc, Miskolc-Egyetemváros
E-mail: matronto@uni-miskolc.hu

J. Varha

Mathematical Faculty of Uzhhorod National University, Uzhhorod, Ukraine
E-mail: iana.varga@uzhnu.edu.ua

We study the non-local problem

$$u'(t) = f(t, u(t)), \ \ t \in [a, b]; \quad \phi(u) = d, \tag{1}$$

where $\phi : C([a, b], \mathbb{R}^n)$ is a vector functional (possibly non-linear), $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function defined on a bounded set and d is a given vector.

In [9], we have suggested an approach to this problem which involves a kind of reduction to a parametrized family of problems with separated conditions

$$u'(t) = f(t, u(t)), \ \ t \in [a, b], \tag{2}$$

$$u(a) = \xi, \quad u(b) = \eta, \tag{3}$$

where $z := \operatorname{col}(z_1, \dots, z_n)$, $\eta := \operatorname{col}(\eta_1, \dots, \eta_n)$ are unknown parameters. The techniques of [9] are based on properties of the iteration sequence $\{u_m(\,\cdot\,, \xi, \eta) : m \geq 0\}$,

$$u_0(t, \xi, \eta) := \Big(1 - \frac{t-a}{b-a}\Big) z + \frac{t-a}{b-a}\, \eta, \tag{4}$$

$$u_m(t, \xi, \eta) := u_0(t, \xi, \eta) + \int_a^t f\big(s, u_{m-1}(s, \xi, \eta)\big)\, ds - \frac{t-a}{b-a} \int_a^b f\big(s, u_{m-1}(s, \xi, \eta)\big)\, ds, \ \ t \in [a, b], \ \ m = 1, 2, \dots . \tag{5}$$

Formulas (4) and (5) are used to compute the corresponding functions explicitly for certain values of m, which, under additional conditions, allows one to prove the solvability of the problem and construct approximate solutions.

The efficiency of application of this approach depends on the complexity of the non-linear terms appearing in (1). If the function f involves transcendental non-linearities with respect the second variable, the explicit computation according to (5) in the general case cannot be carried out due to the impossibility to find the exact values of the corresponding integrals. Here, we show how this difficulty can be overcome using the polynomial interpolation.

At first, we recall some results of the theory of approximations [1, 2, 4]. In a similar situation, we have used these facts in [7].

Denote by P_q the set of all polynomials of degree not higher than q on $[a,b]$. For any continuous $y : [a,b] \to \mathbb{R}$, there exists [2, 7] a unique polynomial $y^q \in H_q$ for which $\|y - y^q\| = \inf\limits_{p\in H_q} \|y - p\| =: E_q(y)$, where $\|\cdot\|$ is the uniform norm in $C([a,b])$. Then y^q is the polynomial of the *best uniform approximation* of y in H_q and the number $E_q(y)$ is called the *error* of the best uniform approximation.

For given continuous function $y : [a,b] \to \mathbb{R}$ and a natural number q, denote by $T_q y$ the Lagrange interpolation polynomial of degree q such that $(T_q y)(t_i) = y(t_i)$, $i = 1, 2, \dots, q+1$, where

$$t_i = \frac{b-a}{2}\cos\frac{(2i-1)\pi}{2(q+1)} + \frac{a+b}{2}\,, \quad i = 1, 2, \dots, q+1, \tag{6}$$

are the Chebyshev nodes translated from $(-1,1)$ to the interval (a,b).

Lemma 1 ([5, p. 18])**.** *For any $q \ge 1$ and a continuous $y : [a,b] \to \mathbb{R}$, the corresponding interpolation polynomial constructed with the Chebyshev nodes admits the estimate*

$$|y(t) - (T_q y)(t)| \le \Big(\frac{2}{\pi}\ln q + 1\Big)E_q(y), \quad t \in [a,b]. \tag{7}$$

Definition 1. Let $y : [a,b] \to \mathbb{R}$ be continuous. The function

$$\delta \longmapsto \omega(y;\delta) := \sup_{t,s\in[a,b]:\ |t-s|\le\delta} |y(t) - y(s)|$$

is called its *modulus of continuity.*

Note that $\delta \mapsto \omega(y;\delta)$ is a continuous non-decreasing function. The function y is uniformly continuous if and only if $\lim\limits_{\delta\to 0}\omega(y;\delta) = 0$ [3, p. 131].

Lemma 2 (Jackson's theorem; [4, p. 22])**.** *If $y \in C([a,b],\mathbb{R})$, $q \ge 1$, then*

$$E_q(y) \le 6\omega\Big(y; \frac{b-a}{2q}\Big). \tag{8}$$

Definition 2. A function $y : [a,b] \to \mathbb{R}$ satisfies the *Dini–Lipschitz condition* [2, p. 50] if

$$\lim_{\delta\to 0}\omega(y;\delta)\ln\delta = 0.$$

It follows from (8) that

$$\lim_{q\to\infty} E_q(y)\ln q = 0 \tag{9}$$

for any y satisfying the Dini–Lipschitz condition. In view of (7), equality (9) ensures the uniform convergence of interpolation polynomials at Chebyshev nodes for this class of functions. In particular, every α-Hölder continuous function $y : [a,b] \to \mathbb{R}$ with $0 < \alpha \le 1$ satisfies the Dini–Lipschitz condition.

Here, we need to construct interpolation polynomials for functions obtained as a result of application of the Nemytskii operator generated by the non-linearity from (1) and defined by the formula

$$(Ny)(t) := f(t, y(t)), \quad t \in [a,b], \tag{10}$$

for any y from $C([a,b],\mathbb{R}^n)$.

Lemma 3. *Let the function $f:[a,b]\times\Omega\to\mathbb{R}^n$, $\Omega\subset\mathbb{R}^n$, satisfy the condition*

$$|f(t,x)-f(s,y)|\le k|t-s|^{\alpha}+K|x-y| \tag{11}$$

for $\{t,s\}\subset[a,b]$, $\{x,y\}\subset\Omega$, where $\alpha\in(0,1]$, $k\in\mathbb{R}^n_+$ and K is an $n\times n$ matrix with non-negative entries. Then, for any Hölder-continuous function $u:[a,b]\to\Omega$, the corresponding function Nu also has this property.

Here and below, the absolute value sign and inequalities between vectors are understood componentwise.

Rewrite (5) in the form

$$u_m(t,\xi,\eta)=u_0(t,\xi,\eta)+\big(\Lambda N u_{m-1}(\,\cdot\,,\xi,\eta)\big)(t),\ \ t\in[a,b],\ \ m=1,2,\dots, \tag{12}$$

where N is the Nemytskii operator (10) and

$$(\Lambda y)(t):=\int\limits_a^t y(s)\,ds-\frac{t-a}{b-a}\int\limits_a^b y(s)\,ds,\ \ t\in[a,b], \tag{13}$$

for any y from $C([a,b],\mathbb{R}^n)$.

Fix a natural number q and extend the notation $T_q y$ for vector-functions by putting $T_q y:=\operatorname{col}(T_q y_1,T_qT_q y_2,\dots,T_q y_n)$ for any $y=(y_i)_{i=1}^n$ from $C([a,b],\mathbb{R}^n)$, where $T_q y_i$ is the qth order interpolation polynomial for y_i constructed with the Chebyshev nodes (6).

Introduce now a modified iteration process keeping formula (4) for $u_0(\,\cdot\,,\xi,\eta)$:

$$v_0^q(\,\cdot\,,\xi,\eta):=u_0(\,\cdot\,,\xi,\eta) \tag{14}$$

and replacing (12) by the formula

$$v_m^q(t,\xi,\eta):=u_0(t,\xi,\eta)+\big(\Lambda T_q N v_{m-1}^q(\,\cdot\,,\xi,\eta)\big)(t),\ \ t\in[a,b],\ \ m\ge 1. \tag{15}$$

For any q, formula (15) defines a vector polynomial $v_m^q(\,\cdot\,,\xi,\eta)$ of degree $q+1$ (in particular, all these functions are continuously differentiable), which, moreover, satisfies the two-point boundary conditions (3). The coefficients of the interpolation polynomials depend on the parameters ξ and η.

Note that, under condition (11), in view of Lemma 3, the function $Nv_{m-1}^q(\,\cdot\,,\xi,\eta))$ appearing in (15) always satisfies the Dini–Lipschitz condition and, therefore, the corresponding interpolation polynomials at Chebyshev nodes uniformly converge to it.

Similarly to (12), functions (15) can be used to study the auxiliary problems (2).

In order to proceed, we introduce some notation. The symbol I_n stands for the unit matrix of dimension n, $r(K)$ denotes a spectral radius of a square matrix K. If $z\in\mathbb{R}^n$ and $\varrho\in\mathbb{R}^n_+$, the componentwise ϱ-neighbourhood of z is defined as $O_\varrho(\xi):=\{\xi\in\mathbb{R}^n:\ |\xi-z|\le\varrho\}$ and, similarly, we put $O_\varrho(\Omega):=\bigcup\limits_{z\in\Omega}O_\varrho(z)$ for any bounded $\Omega\subset\mathbb{R}^n$.

Let $\delta_\Omega(f):=\max\limits_{(t,z)\in[a,b]\times\Omega}f(t,z)-\min\limits_{(t,z)\in[a,b]\times\Omega}f(t,z)$, with the componentwise computation of maxima and minima for vector functions. For $\Omega\subset\mathbb{R}^n$, put $P_{q,\Omega}:=\{u:\ \ u=(u_i)_{i=1}^n,\ u_i\in P_q,\ u([a,b])\subset\Omega\}$. For a given continuous function $f:[a,b]\times\Omega\to\mathbb{R}^n$, set

$$l_{q,\Omega}(f):=\Big(\frac{2}{\pi}\,\ln q+1\Big)\sup_{p\in P_{q,\Omega}}E_q(Np), \tag{16}$$

where N is given by (10) and E_q is the error of the best minimax approximation in P_q. In (16), we use the notation $E_q(u)=\operatorname{col}(E_q(u_1),\dots,E_q(u_n))$ for $u=(u_i)_{i=1}^n$, the least upper bound in (16) is understood componentwise.

Fix certain closed bounded sets D_a, D_b in $\mathbb{R}^n$ and assume that we are looking for solutions u of problem (1) with $u(a) \in D_a$ and $u(b) \in D_b$. Put

$$\Omega := \big\{(1-\theta)\xi + \theta\eta : \ \xi \in D_a, \ \eta \in D_b, \ \theta \in [0,1]\big\}. \tag{17}$$

Theorem 1. *Let there exist a non-negative vector ϱ such that*

$$\varrho \geq \frac{b-a}{4}\,\big(\delta_{O_\varrho(\Omega)}(f) + 2l_{q,O_\varrho(\Omega)}(f)\big). \tag{18}$$

Assume, in addition, that f in (1) *satisfies condition* (11) *on the set $[a,b] \times O_\varrho(\Omega)$ with some k and K and the maximal eigenvalue of K satisfies the inequality*

$$r(K) < \frac{10}{3(b-a)}\,. \tag{19}$$

Then, for all fixed $(\xi,\eta) \in D_a \times D_b$:

1) *For any $m \geq 0$, $q \geq 1$, the function $v_m^q(\,\cdot\,,\xi,\eta)$ is a vector polynomial of degree $q+1$ having values in $O_\varrho(\Omega)$ and satisfying the two-point conditions* (3).

2) *The limits*

$$v_\infty^q(\,\cdot\,,\xi,\eta) := \lim_{m\to\infty} v_m^q(\,\cdot\,,\xi,\eta), \ \ q \geq 1; \quad v_\infty(\,\cdot\,,\xi,\eta) := \lim_{q\to\infty} v_\infty^q(\,\cdot\,,\xi,\eta) \tag{20}$$

   exist uniformly on $[a,b]$. Functions (20) *satisfy conditions* (3).

3) *The estimate*

$$\big|v_\infty(t,\xi,\eta) - v_m^q(t,\xi,\eta)\big| \leq \frac{5}{9}\,\alpha_1(t) K_*^m (1_n - K_*)^{-1} \big(\delta_{O_\varrho(\Omega)}(f) + 2l_{q,O_\varrho(\Omega)}(f)\big)$$

   holds for any $t \in [a,b]$, $m \geq 0$, where $K_* := 3K(b-a)/10$ and

$$\alpha_1(t) = 2(t-a)\Big(1 - \frac{t-a}{b-a}\Big), \ \ t \in [a,b].$$

As follows from [9], the assumptions of Theorem 1, in particular, ensure the uniform convergence of sequence (4), (5) and its limit coincides with $v_\infty(\,\cdot\,,\xi,\eta)$. It is important to point out that, in contrast to formula (12), every component of $v_m^q(\,\cdot\,,\xi,\eta)$, $m \geq 0$, is a *polynomial* of degree $q+1$.

Theorem 2. *Let $(\xi,\eta) \in \Omega$. Under the assumptions of Theorem* 1*, the following two conditions are equivalent:*

1) *The function $u := v_\infty(\,\cdot\,,\xi,\eta) : [a,b] \to \mathbb{R}^n$ is a continuously differentiable solution of problem* (1) *such that $u(a) \in D_a$, $u(b) \in D_b$, and $u([a,b]) \subset O_\varrho(\Omega)$.*

2) *The pair (ξ,η) satisfies the system of $2n$ determining equations*

$$\eta - \xi = \int_a^b f\big(s, v_\infty(s,\xi,\eta)\big)\,ds, \quad \phi(v_\infty(\,\cdot\,,\xi,\eta)) = d. \tag{21}$$

The determining system (21) can be investigated by using properties of its approximate version

$$\eta - \xi = \int_a^b f\big(s, v_m^q(s, \xi, \eta)\big)\, ds, \quad \phi(v_m^q(\,\cdot\,, \xi, \eta)) = d, \tag{22}$$

where m and q are fixed. The solvability analysis based on properties of equations (22) can be carried out by analogy to [6, 10].

Although (18) is more restrictive than the corresponding condition from [9]

$$\varrho \geq \frac{b-a}{4}\, \delta_{O_\varrho(\Omega)}(f), \tag{23}$$

one can note that, by virtue of (16) and Lemmas 1–3, $\lim\limits_{q\to\infty} l_{q,O_\varrho(\Omega)}(f) = 0$. Furthermore, both (18) and (23) can be relaxed (and, in fact, the resulting conditions eventually fulfilled) by using interval divisions similarly to [8]. The same observation can be made on condition (19), which, in particular, after one division in the ratio $1:2$, is replaced by the condition

$$r(K) < \frac{20}{3(b-a)}\,.$$

As an example of application of the approach based on the polynomial approximations (14), (15), consider the system of differential equations

$$\begin{aligned} u_1'(t) &= u_1(t)u_2(t), \\ u_2'(t) &= -\ln(2u_1(t)), \ \ t \in \Big[0, \frac{\pi}{4}\Big], \end{aligned} \tag{24}$$

with the non-linear two-point boundary conditions

$$u_1(0) - \Big(u_2\Big(\frac{\pi}{4}\Big)\Big)^2 = \frac{3}{8}, \quad u_1(0)u_2\Big(\frac{\pi}{4}\Big) = \frac{\sqrt{2}}{8}\,. \tag{25}$$

Choose the subsets D_a and D_b, where one looks for the values $u(a)$ and $u(b)$, e.g., as follows:

$$D_a = \big\{(u_1, u_2):\ 0.45 \leq u_1 \leq 0.75,\ 0.4 \leq u_2 \leq 0.55\big\}, \ \ D_b = D_a.$$

In this case, set (17) has the form $\Omega = D_a = D_b$. Putting $\varrho = \operatorname{col}(0.2, 0.35)$, we get

$$O_\varrho(\Omega) = \big\{(u_1, u_2):\ 0.25 \leq u_1 \leq 0.95,\ 0.05 \leq u_2 \leq 0.9\big\}.$$

A direct computation shows that the conditions of Theorem 1 are satisfied for q large enough. By solving the polynomial approximate determining equations (22), we obtain the numerical values of parameters ξ_1, ξ_2, η_1, η_2, which determine the polynomial approximate solution of the given problem (24), (25). In particular, for $q = 4$ and $m = 7$, the approximate solution $u_7^4 = \operatorname{col}(u_{71}^4, u_{72}^4)$ is a vector polynomial of degree 5,

$$\begin{aligned} u_{71}^4(t) &\approx 0.00456\, t^5 - 0.02668\, t^4 - 0.02838\, t^3 + 0.06195\, t^2 + 0.24987\, t + 0.5, \\ u_{72}^4(t) &\approx 0.49982 - 0.0017\, t^5 + 0.02231\, t^4 - 0.00062\, t^3 - 0.24956\, t^2 + 0.49982. \end{aligned}$$

A comparison with the exact solution

$$u_1(t) = \frac{1}{2}\, \exp\Big(\frac{1}{2}\, \sin t\Big), \quad u_2(t) = \frac{1}{2}\, \cos t$$

shows a high degree of accuracy of the approximate polynomial solution.

References

[1] V. L. Goncharov, *Theory of Interpolation and Approximation of Functions.* (Russian) 2d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954.

[2] I. P. Natanson, *Constructive Function Theory.* Vol. III. *Interpolation and Approximation Qquadratures.* Translated from the Russian by John R. Schulenberger Frederick Ungar Publishing Co., New York, 1965.

[3] G. M. Phillips, *Interpolation and Approximation by Polynomials.* CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 14. Springer-Verlag, New York, 2003.

[4] T. J. Rivlin, *An Introduction to the Approximation of Functions.* Blaisdell Publishing Co. Ginn and Co., Waltham, Mass.-Toronto, Ont.-London, 1969.

[5] T. J. Rivlin, The Chebyshev Polynomials. Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York–London–Sydney, 1974.

[6] A. Rontó and M. Rontó, Existence results for three-point boundary value problems for systems of linear functional differential equations. *Carpathian J. Math.* **28** (2012), no. 1, 163–182.

[7] A. Rontó, M. Rontó, G. Holubová and P. Nečesal, Numerical-analytic technique for investigation of solutions of some nonlinear equations with Dirichlet conditions. *Bound. Value Probl.* **2011**, 2011:58, 20 pp.

[8] A. Rontó, M. Rontó, and N. Shchobak, Notes on interval halving procedure for periodic and two-point problems. *Bound. Value Probl.* **2014**, 2014:164, 20 pp.

[9] A. Rontó, M. Rontó and J. Varha, A new approach to non-local boundary value problems for ordinary differential systems. *Appl. Math. Comput.* **250** (2015), 689–700.

[10] M. Rontó and Y. Varha, Constructive existence analysis of solutions of non-linear integral boundary value problems. *Miskolc Math. Notes* **15** (2014), no. 2, 725–742.