Antiperiodic Impulsive Problem Via Distributional Differential Equation

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1 Introduction

Analytical results presented here are based on a common research with Jan Tomeček. We focus our attention to problems where the evolution of systems is affected by rapid changes which is modelled by means of differential equations with impulses. Let us stress that abrupt changes of solutions of impulsive problems imply that such solutions do not preserve the basic properties which are associated with non-impulsive problems.

We work with a finite number $m \in \mathbb{N}$ of impulses on the compact interval $[0,T] \subset \mathbb{R}$. Most papers deal with *fixed-time* impulses where the moments of impulses

$$0 < t_1 < t_2 < \dots < t_m < T$$

are fixed and known before. This is a special case of so called *state-dependent* impulses where the impulse moments depend on a solution of a differential equations and different solutions can have different moments of jumps. We present two ways of determining the impulse dependence on the solution:

- Let $\tau_1, \dots, \tau_m$ be functionals defined on a suitable functional space X and having values in $(0,T)$. Then the impulse moments $t_1, \dots, t_m$ are given as

$$t_i = \tau_i(x) \in (0,T), \ \ x \in X, \ \ i = 1, \dots, m.$$

- Let $\gamma_1, \dots, \gamma_m$ be functions (barriers) defined on a suitable interval $[a,b] \subset \mathbb{R}$ and having values in $(0,T)$. Then the impulse moments $t_1, \dots, t_m$ are given as

$$t_i = \gamma_i(x(t_i)) \in (0,T), \ \ x \in X, \ \ i = 1, \dots, m.$$

  In order to get the desired number of impulse points in this case it is necessary to impose additional conditions (transversality conditions) on $\gamma_1, \dots, \gamma_m$.

2 Periodic problems

A lot of papers studying impulsive periodic problems are population or epidemic models. Differential equations in these models have mostly a form of *autonomous* planar differential systems. However, there are a few existence results for non-autonomous periodic problems with state-dependent impulses:

- The first attempts can be seen in the monographs [1] and [11] investigating periodic solutions of quasilinear systems with state-dependent impulses.

- One of the first results that are trackable via Scopus is obtained by Bajo and Liz [2], where a scalar nonlinear first order differential equation is studied under the assumptions of the existence and uniqueness of a solution of the corresponding initial value problem with state-dependent impulses, and of the existence of lower and upper solutions of the periodic problem with state-dependent impulses. Their method of proof is based on a fixed point theorem for a Poincaré operator.

- A generalization to a system is done by Frigon and O'Regan [8] under the assumption that there exists a solution tube to the problem. They applied a fixed point theorem to a multi-valued Poincaré operator.

- Further interesting result is reached by Domoshnitsky, Drakhlin and Litsyn in [7]. They transformed a linear system with delay and state-dependent impulses to a system with fixed-time impulses and then they proved the existence of positive periodic solutions.

- Recently, Tomeček [12] proved the existence of a periodic solution to a nonlinear second order differential equation with ϕ-Laplacian and state-dependent impulses via lower and upper solutions method.

All the above problems have a *"classical"* formulation in which impulse conditions are given out of a differential equation. Let us demonstrate it on the van der Pol equation

$$x'(t) = y(t), \quad y'(t) = \mu\Big(x(t) - \frac{x^3(t)}{3}\Big)' - x(t) + f(t) \ \text{ for a.e. } \ t \in [0,T], \tag{1}$$

with the state-dependent impulse conditions

$$\triangle y(\tau_i(x)) = \mathcal{J}_i(x), \ \ i = 1, \dots, m, \tag{2}$$

where T, $\mu > 0$, $m \in \mathbb{N}$, τ_i, $\mathcal{J}_i$, $i = 1, \dots, m$, are functionals defined on the set of T-periodic functions of bounded variation and f is T-periodic Lebesgue integrable on $[0,T]$. Here x' and y' denote the classical derivatives of the functions x and y, respectively, $\triangle y(t) = y(t+) - y(t-)$.

Another possible formulation of the T-periodic problem with state-dependent impulses at the points $\tau_i(x) \in (0,T)$ can be written in the form of the *distributional* differential equation

$$D^2 z = \mu D\Big(z - \frac{z^3}{3}\Big) - z + f + \frac{1}{T}\sum_{i=1}^{m} \mathcal{J}_i(z)\delta_{\tau_i(z)}, \tag{3}$$

where Dz denotes the distributional derivative of a T-periodic function z of bounded variation and $\delta_{\tau_i(z)}$, $i = 1, \dots, m$, are the Dirac T-periodic distributions which involve impulses at the state-dependent moments $\tau_i(z)$, $i = 1, \dots, m$.

Results on the existence of periodic solutions to distributional equations of the type (3) have been reached by Belley, Virgilio and Guen in [4–6].

3 Antiperiodic problems

The study of antiperiodic solutions is closely related to the study of periodic solutions and their existence plays an important role in characterizing the behaviour of nonlinear differential equations. First order differential systems with antiperiodic conditions can describe neural networks and second order differential equations can serve as physical models, for example: Rayleigh equation (acoustics), Duffing, Liénard or van der Pol equations (oscillation theory).

In the study of T-antiperiodic solutions we work with functional spaces defined below which consist of *real-valued* $2T$*-periodic functions*: NBV is the space of functions of bounded variation normalized in the sense that $x(t)=\frac{1}{2}\,(x(t+)+x(t-))$, $\widetilde{\mathrm{NBV}}$ represents the Banach space of functions $x\in\mathrm{NBV}$ having zero mean value, which is equipped with the norm equal to the total variation $\mathrm{var}(x)$, C^∞ is the classical Fréchet space of functions having derivative of an arbitrary order, for finite $\Sigma\subset[0,2T)$ we denote by PAC_Σ the set of all functions $x\in\mathrm{NBV}$ such that $x\in\mathrm{AC}(J)$ for each interval $J\subset[0,2T]$ for which $\Sigma\cap J=\varnothing$, $\widetilde{\mathrm{AC}}=\mathrm{AC}\cap\widetilde{\mathrm{NBV}}$; for finite $\Sigma\subset[0,2T)$ we denote $\widetilde{\mathrm{PAC}}_\Sigma=\mathrm{PAC}_\Sigma\cap\widetilde{\mathrm{NBV}}$.

The first result about the existence and uniqueness of antiperiodic solutions of the distributional Liénard equation with state-dependent imupulses has been reached by Belley and Bondo in [3].

- In order to study T-antiperiodic solutions for the classical differential Liénard equation (1) with state-dependent impulses (2) we assume that f in (1) is T-antiperiodic and that the condition
$$\tau_i(x)\neq\tau_j(x)\ \text{ for all }\ i,j=1,\dots,m,\ \ i\neq j,\ \ x\in\widetilde{\mathrm{AC}}$$
is fulfilled.

- For $x\in\widetilde{\mathrm{AC}}$ we denote the set
$$\Sigma_x:=\big\{\tau_1(x),\dots,\tau_m(x),\tau_1(x)+T,\dots,\tau_m(x)+T\big\},$$
and say that the couple $(x,y)\in\widetilde{\mathrm{AC}}\times\widetilde{\mathrm{PAC}}_{\Sigma_x}$ is a *solution* of the impulsive problem (1), (2) if it satisfies the differential equation (1) and the impulse conditions (2). Such solution (x,y) is called *antiperiodic* if
$$x(0)=-x(T),\quad y(0)=-y(T).$$

- Motivated by [3] we construct the following distributional van der Pol equation
$$D^2z=\mu D\Big(z-\frac{z^3}{3}\Big)-z+f+\frac{1}{2T}\sum_{i=1}^m\mathcal{J}_i(z)\varepsilon_{\tau_i(z)}. \tag{4}$$
The Dirac T-periodic distribution δ_τ from (3) is replaced by the T-antiperiodic distribution $\varepsilon_\tau:=\delta_\tau-\mathbb{T}_T\delta_\tau$ in (4). Here $\mathbb{T}_T$ means the translation operator.

- We say that a function $z\in\widetilde{\mathrm{NBV}}$ is a *solution* of the distributional equation (4) if
$$\langle D^2z,\varphi\rangle=\Big\langle\mu D\Big(z-\frac{z^3}{3}\Big)-z+f+\frac{1}{2T}\sum_{i=1}^m\mathcal{J}_i(z)\varepsilon_{\tau_i(z)},\varphi\Big\rangle\ \text{ for every }\ \varphi\in\mathrm{C}^\infty. \tag{5}$$

By means of the method of apriori estimates and the Schauder fixed point theorem we get a new existence result for (4). Then, using an equivalence between classical and distributional problems which we proved in [9], we get the first result about the existence of a T-antiperiodic solution of equation (1) with state-dependent impulses (2):

Theorem 3.1. *Assume* $T\in(0,\sqrt{3})$ *and*

1. f *is* T*-antiperiodic and Lebesgue integrable on* $[0,T]$*;*
2. $\tau_1,\dots,\tau_m$ *are continuous with values in* $(0,T)$*;*
3. *if* $i\neq j$*, then* $\tau_i(x)\neq\tau_j(x)$ *for each* $x\in\widetilde{\mathrm{AC}}$*;*

4. $\mathcal{J}_1, \dots, \mathcal{J}_m$ *are continuous and bounded.*

Then there exists $\mu_0 > 0$ *such that for each* $\mu \in (0, \mu_0]$ *the problem* (1), (2) *has a* T*-antiperiodic solution.*

Theorem 3.1 and its generalizations are published in [10] where the optimal value of μ_0 is also specified.

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