On Stability Properties of Global Attractors of Impulsive Infinite-Dimensional Systems

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One of the most popular mathematical approaches to describe the evolutionary processes with instantaneous changes is the theory of impulsive differential equations. Today due to the works of A. M. Samoĭlenko, M. O. Perestyuk [11], R. Lakshmikantham, D. Bainov [7] and many other mathematicians the theory of impulsive systems became a separate branch of the general theory of differential equations. An important subclass of systems with impulsive perturbations is impulsive (or discontinuous) dynamical systems, which are described by an autonomous evolutionary system, which trajectories has impulsive perturbations at moments of intersection with fixed subset of the phase space (impulsive set). Unlike systems with impulsive perturbations at fixed moments of time, the construction of a qualitative theory for impulsive dynamic systems is far from complete understanding. Various aspects of qualitative theory of such systems in the finite-dimensional case was studied in [1, 3, 6, 8, 9]. For infinite-dimensional dissipative systems one of the most powerful tools of investigation of the qualitative behavior of solutions is the theory of global attractors [12]. In [10] the theory of global attractors was used for investigation of impulsive systems with fixed moments of impulsive perturbations. However, the transferring of the basic constructions of this theory to impulsive dynamical systems has a fundamental problem – the absence of continuous dependence of the initial data. This requires a new concept for the global attractor, and for its main characteristics (invariance, stability, robustness). One of possible approaches was proposed in [5] and was based on the concept of a uniform attractor for non-autonomous systems – a compact minimal uniformly-attracting set. The absence an invariance condition in such definition allowed not to use strict conditions for the behavior of trajectories in the neighborhood of the impulsive set (see [2]) and obtain results about the existence and properties of the attractor for weakly nonlinear impulse-perturbed equations. Further in [4], this approach was extended to other classes of impulsive systems, in particular those for which the uniqueness of the solution of Cauchy problem is not fulfilled. The aim of this work is to study stability concept of the global attractor of the impulsive dynamical system and apply obtained results to a weakly nonlinear impulsive system.

Let $G : \mathbb{R}_+ \times X \to R$ be a semigroup, which is given in the normed space X (it is not necessarily continuous), $\beta(X)$ be a set of all nonempty bounded subsets of X.

Definition ([5])**.** A compact set $\Theta \subset X$ is called a global attractor of G if

1) Θ is uniformly attracting set, i.e.,

$$\forall\, B \in \beta(X) \quad \operatorname{dist}(G(t,B),\Theta) \longrightarrow 0, \ \ t \to \infty;$$

2) Θ is minimal among closed uniformly attracting sets.

If global attractor exists, then it is unique. If G has a global attractor in a classical sense [12], i.e. if there exists a compact set $A \subset X$, which satisfies 1) and it is invariant ($G(t,A) = A \ \forall\, t \geq 0$), then A satisfies this definition. On the contrary, it is true that $G(t,\,\cdot\,)$ is continuous. It means that if Θ is global attractor of G and the following condition is fulfilled:

$$\forall\, t \geq 0 \text{ a map } x \to G(t,x) \text{ is continuous,} \tag{1}$$

then Θ is a global attractor of G in a classical sense, in particular

$$\forall\, t \geq 0 \quad \Theta = G(t, \Theta).$$

That is why, in impulsive (discontinuous) dynamical systems, where the semigroup G does not usually satisfy the condition (1), it's better to use this definition.

Another advantage of this definition is the following criterion:

Lemma 1 ([10]). *Assume that G satisfies dissipativity condition:*

$$\exists\, B_0 \in \beta(X) \ \ \forall\, B \in \beta(X) \ \ \exists\, T = T(B) \ \ \forall\, t \geq T \quad G(t, B) \subset B_0.$$

Then G has global attractor Θ if and only if, when G is asymtotically compact, i.e., $\forall\,\{x_n\} \in \beta(X)$ $\forall\,\{t_n \nearrow \infty\}$ sequence $\{G(t_n, x_n)\}$ is precompact. Moreover,

$$\Theta = \omega(B_0) := \bigcap_{\tau>0} \overline{\bigcup_{t \geq \tau} G(t, B_0)}.$$

Using this criterion, in [5], classes of impulsive infinite-dimensional dissipative problems, which have a global attractor were identified. In particular, it was shown that for sufficiently small $\varepsilon > 0$, solutions of the impulsive problem

$$\begin{aligned} &\frac{\partial y}{\partial t} = \Delta y - \varepsilon f(y), \ \ t > 0, \ \ x \in \Omega, \\ &y\big|_{\partial\Omega} = 0, \end{aligned} \tag{2}$$

where $\Omega \subset \mathbb{R}^p$ is bounded domain, f – continuously-differentiable function, $\forall\, y \in R$ $f'(y) \geq -C$, $|f(y)| \leq C$, in phase space $X = L^2(\Omega)$ with impulsive set

$$M = \big\{ y \in X \ | \quad (y, \psi_1) = a \big\} \tag{3}$$

and impulsive map $I : M \mapsto X$

$$\text{for } \ y = a\psi_1 + \sum_{i=2}^{\infty} c_i \psi_i, \quad Iy = (1+\mu) a\psi_1 + \sum_{i=2}^{\infty} c_i \psi_i, \tag{4}$$

generate the semigroup G_ε, which has a global attractor Θ_ε, moreover,

$$\operatorname{dist}(\Theta_\varepsilon, \Theta_0) \longrightarrow 0, \ \ \varepsilon \to 0,$$

where

$$\Theta_0 = \bigcup_{t \in [0, \ln(1+\mu)]} \big\{ (1+\mu) a e^{-t} \psi_1 \big\} \cup \{0\} \tag{5}$$

is a global attractor of G_0, generated by (2)–(4), where $\varepsilon = 0$.

In case $\varepsilon = 0$ the set Θ_0 has a non-empty intersection with an impulsive set M is not invariant regarding the G_0. However, for $\Theta_0 \setminus M$ we obtain the next equality:

$$\forall\, t \geq 0 \quad G_0(t, \Theta_0 \setminus M) = \Theta_0 \setminus M. \tag{6}$$

This provides the basis to analyze the invariance and stability of the set $\Theta_\varepsilon \setminus M$ for the impulsive dynamical system G_ε with the global attractor Θ_ε. The next well-known result establishes the connection between different definitions of the stability of compact sets regarding the semigroup G.

Lemma 2. *If $A \subset X$ is compact and the condition is fulfilled:*

$$\forall\, x_n \to x \in A \ \ \forall\, t_n \geq 0 \quad \{G(t_n, x_n)\}\text{-precompact}, \tag{7}$$

then the following properties are equivalent:

1) $\forall\, \varepsilon > 0 \ \ \forall\, x \in A \ \ \exists\, \delta > 0 \ \ \forall\, t \geq 0$
$$G(t, O_\delta(x)) \subset O_\varepsilon(A);$$

2) $\forall\, \varepsilon > 0 \ \ \exists\, \delta > 0 \ \ \forall\, t \geq 0$
$$G(t, O_\delta(A)) \subset O_\varepsilon(A);$$

3) $\forall\, x \in A \ \ \forall\, y \notin A \ \ \exists\, \delta > 0 \ \ \forall\, t \geq 0$
$$G(t, O_\delta(x)) \cap O_\delta(y) = \varnothing;$$

4) $A = D^+(A) := \bigcup\limits_{x \in A} \big\{ y \mid \ \ y = \lim G(t_n, x_n), \ \ x_n \to x, \ \ t_n \geq 0 \big\}.$

Remark. Because of the construction $A \subset D^+(A)$, property 4) is equivalent to an embedding $D^+(A) \subset A$.

The following result is easily obtained by contradiction.

Lemma 3. *If Θ is global attractor of G and the condition is fulfilled*

$$\forall\, x_n \to x \in \Theta \ \ \forall\, t_n \to t \geq 0 \quad G(t_n, x_n) \longrightarrow G(t, x), \tag{8}$$

then Θ is stable in the sense of 1)–4).

The condition (8) is crucial. Its failure leads to the fact that the attractor Θ may not be stable in any sense of 1)–4). For example, for a semigroup G_0 attractor Θ_0, which is given by (5), does not satisfy 1)–4). Unfortunately, the same thing holds for an invariant set $\Theta_0 \setminus M$. However, it is easy to see that for G_0
$$D^+(\Theta_0 \setminus M) \subset \overline{\Theta_0 \setminus M}. \tag{9}$$

The main aim of this work is to prove that the properties (6), (9) are also fulfilled for a weakly nonlinear case.

Theorem. *For sufficiently small $\varepsilon > 0$ global attractor Θ_ε of impulsive dynamical system G_ε, which is generated by problem* (2)–(4)*, is invariant and stable in the following sense:*

$$\forall\, t \geq 0 \quad \Theta_\varepsilon \setminus M = G_\varepsilon(t, \Theta_\varepsilon \setminus M),$$
$$\Theta_\varepsilon = \overline{\Theta_\varepsilon \setminus M},$$
$$D^+(\Theta_\varepsilon \setminus M) \subset \overline{\Theta_\varepsilon \setminus M}.$$

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