The Periodic Type Problem for the Second Order Integro Differential Equations

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Consider on the interval $I = [0, \omega]$ the second order linear integro differential equation

$$u''(t) = \int\limits_0^\omega p(t,s)u(\tau(s))\,ds + q(t), \tag{1}$$

and the nonlinear functional differential equation

$$u''(t) = F(u)(t) + f(t), \tag{2}$$

with the periodic type two-point boundary value conditions

$$u^{(i-1)}(\omega) - u^{(i-1)}(0) = c_i \ \ (i = 1, 2), \tag{3}$$

where $c_1, c_2 \in R$, $p \in L_\infty(I^2, R)$, $F : C(I,R) \to L(I,R)$ is a continuous operator, $\tau : I \to I$ is a measurable function, and $q \in L(I,R)$.

By a solution of problem (1), (3) we understand a function $u \in \widetilde{C}\,'(I,R)$ which satisfies equation (1) almost everywhere on I and satisfies conditions (3).

Throughout the paper we use the following notations.

R is the set of all real numbers, $R_+ = [0, +\infty[$;

$C(I;R)$ is the Banach space of continuous functions $u : I \to R$ with the norm $\|u\|_C = \max\{|u(t)| : t \in I\}$;

$C\,'(I;R)$ is the Banach space of functions $u : I \to R$ which are continuous together with their first derivatives with the norm $\|u\|_{C'} = \max\{|u(t)| + |u'(t)| :\ t \in I\}$;

$\widetilde{C}\,'(I;R)$ is the set of functions $u : I \to R$ which are absolutely continuous together with their first derivatives;

$L(I;R)$ is the Banach space of the Lebesgue integrable functions $p : I \to R$ with the norm $\|p\|_L = \int\limits_a^b |p(s)|ds$;

$L_\infty(I,R)$ is the space of essentially bounded functions $p : I \to R$ with the norm $\|p\|_\infty = \operatorname{ess\,sup}\{|p(t)| :\ t \in I\}$;

$L_\infty(I^2,R)$ is the set of such functions $p : I^2 \to R$ that for arbitrary $y \in L_\infty(I,R)$ and fixed $t \in I$, $p(t,\,\cdot\,), y(\,\cdot\,) \in L(I,\,R)$, and

$$\int\limits_0^\omega p(\,\cdot\,,s)y(s)\,ds \in L_\infty(I,\,R);$$

For arbitrary $p \in L_\infty(I^2,R)$ and measurable $\tau : I \to I$ we will use the notation:

$$\ell(p,\tau) = \frac{2\pi}{\omega}\bigg(\int\limits_0^\omega\int\limits_0^\omega |p(\xi,\,s)|\,|\tau(s) - \xi|\,ds\,d\xi\bigg)^{1/2}.$$

Definition 1. Let $\sigma \in \{-1, 1\}$. We say that the function $h \in L_\infty(I^2, R)$ belongs to the set $K^\sigma_{I,\tau}$ if $h(t,\, s) \geq 0$ and for an arbitrary function $p \in L_\infty(I^2, R)$ such that

$$0 \leq \sigma p(t,s) \leq h(t,s), \quad \int\limits_0^\omega p(t,\xi)\,d\xi \not\equiv 0 \ \text{ for } \ (t,s) \in I^2,$$

the homogeneous problem

$$v''(t) = \int\limits_0^\omega p(t,s)v(\tau(s))\,ds,$$
$$v^{(i-1)}(\omega) - v^{(i-1)}(0) = 0 \ \ (i = 1, 2),$$

has no nontrivial solution.

Main results

Proposition 1. *Let $\sigma \in \{-1,\, 1\}$,*

$$h(t,s) \geq 0, \quad \int\limits_0^\omega h(t,\xi)\,d\xi \not\equiv 0 \ \textit{ for } \ (t,\, s) \in I^2,$$

and for almost all $t \in I$ the inequality

$$\frac{(1-\sigma)}{2}\int\limits_0^\omega h(t,s)\,ds + \ell(h,\tau)\Big(\int\limits_0^\omega h(t,s)\,ds\Big)^{1/2} < \Big(\frac{2\pi}{\omega}\Big)^2$$

hold. Then

$$h \in K^\sigma_{I,\tau}.$$

Theorem 1. *Let $\sigma \in \{-1,\, 1\}$,*

$$\sigma p(t,s) \geq 0, \quad \int\limits_0^\omega p(t,\xi)\,d\xi \not\equiv 0 \ \textit{ for } \ (t,s) \in I^2,$$

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hold. Then problem $(1), (3)$ *is uniquely solvable.*

On the basis of Proposition 1 we can prove the necessary conditions for solvability and unique solvability of nonlinear problem $(2), (3)$. First for arbitrary $h \in L_\infty(I^2, R)$ and $r > 0$ introduce the set $U_{p,r}$ and the operator $\psi_{h,\tau}(x) : C(I,R) \to L(I,R)$ by the equalities

$$U_{I,r} = \bigg\{ x \in C'(I,R) :\ x^{(i-1)}(\omega) - x^{(i-1)}(\omega) = c_i \ (i = 1, 2), \ \bigg| \int\limits_0^\omega \int\limits_0^\omega h(\xi,s)x(\tau(s))\,ds\,d\xi \bigg| \geq r \bigg\},$$

$$\psi_{p,\tau}(x)(t)=\begin{cases}1 & \text{for } \int\limits_0^\omega h(t,s)x(\tau(s))\,ds>0,\\ 0 & \text{for } \int\limits_0^\omega h(t,s)x(\tau(s))\,ds=0,\\ -1 & \text{for } \int\limits_0^\omega h(t,s)x(\tau(s))\,ds<0,\end{cases}$$

and the class $K_{p,\tau}(C,L)$ of the operators by the next definition.

Definition 2. Let $p\in L_\infty(I^2,R)$ and $\tau:I\to I$ be measurable function, then we say that $F\in K_{p,\tau}(C,L)$ if $F:C(I,R)\to L(I,R)$ is continuous operator and for arbitrary $r>0$

$$\sup\bigg\{|F(u)(t)|:\ \bigg|\int\limits_0^\omega\int\limits_0^\omega h(\xi,s)u(\tau(s))\,ds\,d\xi\bigg|\le r\bigg\}\in L(I,\,R).$$

Then the next theorem is true.

Theorem 2. *Let $F\in K_{h,\tau}(C,L)$, there exist numbers $\sigma\in\{-1,1\}$, $r_0>0$, measurable $\tau:I\to I$ and functions $g,g_0\in L(I,R_+)$, and $h\in K^\sigma_{I,\tau}$ such that for arbitrary $x\in U_{h,r_0}$ on I the next conditions hold*

$$g_0(t)\le\sigma F(x)(t)\psi_{h,\tau}(x)(t)\le\bigg|\int\limits_0^\omega h(t,s)x(\tau(s))\,ds\bigg|+g(t)$$

if $\psi_{h,\tau}(x)(t)\neq0$, and

$$F(x)(t)=0\ \text{ if }\ \psi_{h,\tau}(x)(t)=0.$$

Let, moreover,

$$\bigg|\int\limits_0^\omega f(s)\,ds\bigg|\le\int\limits_0^\omega g_0(s)\,ds-|c_2|. \tag{4}$$

Then problem (2), (3) has at least one solution.

From this theorem for the equation

$$u''(t)=f_0\bigg(t,\int\limits_0^\omega p(t,s)u(\tau(s))\,ds\bigg)+f(t), \tag{5}$$

where the function $f_0:I^2\to R$ is from the Carathéeodory's class, $f_0(t,0)\equiv0$, $p\in L_\infty(I^2,R)$, and $\tau:I\to I$ is a measurable function, it follows

Corollary 1. *Let there exist numbers $\sigma\in\{-1,1\}$, $r_0>0$, and functions $w,g,g_0\in L(I,\,R_+)$ such that $w(t)>0$, $\sigma,w,p\in K^\sigma_{I,\tau}$,*

$$g_0(t)\le f_0(t,x)\operatorname{sgn}x\le w(t)|x|+g(t)\ \text{ for }\ |x|>r_0,\ \ t\in I,$$

and inequality (4) holds. Then problem (5), (3) has at least one solution.

Remark. Inequality (4) in Theorem 2 (Corollary 1) cannot be replaced by the inequality

$$\Big|\int_0^\omega f(s)\,ds\Big| \le \int_0^\omega g_0(s)\,ds - |c_2| + \varepsilon \tag{6}$$

no matter how small $\varepsilon > 0$ would be. Indeed, if $F \equiv 0$, $f(t) \equiv \frac{\varepsilon}{\omega}$, $g_0 \equiv g \equiv 0$, $c_2 = 0$, and

$$h(t,s) \equiv \Big(\frac{2\pi}{\omega}\Big)^2 \frac{1-\varepsilon}{\ell(1,\tau)\sqrt{\omega}},$$

then instead of (4) inequality (6) holds and all other conditions of Theorem 2 (Corollary 1) are fulfilled. Nevertheless, in that case problem (2), (3) is not solvable.

Theorem 3. *Let $F \in K_{h,\tau}(C,L)$, there exist a number $\sigma \in \{-1,1\}$, a continuous functional $\eta : C(I,R) \to R_+$, measurable $\tau : I \to I$, and functions $h_0 \in L(I,R_+)$, $h \in K^\sigma_{I,\tau}$ such that for arbitrary $x, y \in U_{h,0}$ on I the next conditions hold*

$$h_0(t)\eta(x-y) \le \sigma(F(x)(t) - F(y)(t))\psi_{h,\tau}(x-g)(t) \le \Big|\int_0^\omega h(t,s)\big(x(\tau(s)) - y(\tau(s))\big)\,ds\Big|$$

if $\psi_{h,\tau}(x-y)(t) \neq 0$, and

$$F(x-y)(t) = F(x)(t) - F(y)(t) = 0$$

if $\psi_{h,\tau}(x-y)(t) = 0$, where

$$h_0(t) \gneqq 0 \ \text{ for } \ t \in I, \quad \eta(z) > 0 \ \text{ if } \ \min_{t\in I}\{|z(t)|\} > 0.$$

Let, moreover, there exist a number $r_0 > 0$ such that condition (4) holds, where

$$g_0(t) = h_0(t)\min\left\{\eta(z) : \ \|z\|_C \ge \frac{r_0}{\int_0^\omega\int_0^\omega h(\xi,s)\,ds\,d\xi}\right\}.$$

Then problem (2), (3) is uniquely solvable.

As examples we consider the differential equations

$$u''(t) = \frac{\sigma(1 + |\sin u(\tau_0(t))|)}{2}\int_0^\omega h(t,s)u(\tau(s))\,ds + f_0(t) \tag{7}$$

and

$$u''(t) = \frac{\sigma\int_0^\omega h(t,s)u(\tau(s))\,ds}{(1 + \big|\int_0^\omega h(t,s)u(\tau(s))\,ds\big|)^\alpha} + f_0(t), \tag{8}$$

where $\sigma \in \{-1,1\}$, $\alpha \in (0,1]$, $h \in K^\sigma_{I,\tau}$, and $\int_0^\omega f_0(s)\,ds = 0$. Then in view of Theorem 2 with $r_0 > |c_2|\,\big|\int_0^\omega h(t,s)u(\tau(s))\,ds\big|^{-1}$ (Theorem 3) problem (7), (3) (problem (8), (3)) is solvable (uniquely solvable).

References

[1] E. I. Bravyi, On the best constants in the solvability conditions for the periodic boundary value problem for higher-order functional differential equations. (Russian) *Differ. Uravn.* **48** (2012), no. 6, 773–780; translation in *Differ. Equ.* **48** (2012), no. 6, 779–786.

[2] R. Hakl and S. Mukhigulashvili, A periodic boundary value problem for functional differential equations of higher order. *Georgian Math. J.* **16** (2009), no. 4, 651–665.

[3] I. T. Kiguradze, Boundary value problems for systems of ordinary differential equations. (Russian) Translated in *J. Soviet Math.* **43** (1988), no. 2, 2259–2339. Itogi Nauki i Tekhniki, *Current problems in mathematics. Newest results, Vol. 30 (Russian)*, 3–103, 204, *Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow*, 1987.

[4] S. Mukhigulashvili, On periodic solutions of second order functional differential equations. *Ital. J. Pure Appl. Math.*, no. 20 (2006), 29–50.