On Coefficient Perturbation Classes with Degeneracies

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Consider the linear differential system

$$\dot{x} = A(t)x, \;\; x \in \mathbb{R}^n, \;\; t \geq 0, \tag{1}$$

with a piecewise continuous bounded coefficient matrix A such that $\|A(t)\| \leq a < +\infty$ for all $t \geq 0$. Together with system (1), consider the perturbed system

$$\dot{y} = A(t)y + Q(t)y, \;\; y \in \mathbb{R}^n, \;\; t \geq 0, \tag{2}$$

with a piecewise continuous bounded perturbation matrix Q. For the higher exponent of system (2), we use the notation $\lambda_n(A+Q)$. By $\mathbb{R}^{n\times n}$ we denote the set of all real $n \times n$-matrices with the spectral norm $\|\cdot\|$.

Let $\mathfrak{M}$ be a class of perturbations. The number $\Lambda(\mathfrak{M}) := \sup\{\lambda_n(A+Q) : \;\; Q \in \mathfrak{M}\}$ is an important asymptotic characteristics for system (1). Many authors investigated how to find $\Lambda(\mathfrak{M})$ for various $\mathfrak{M}$, see, e.g., the monographs [3, p. 157], [7, p. 39], the review [5], and the papers [1, 2, 4, 6, 8–16], where the following $\mathfrak{M}$ are considered:

- vanishing at infinity perturbations [15]

$$Q(t) \to 0, \;\; t \to +\infty;$$

- exponentially small perturbations [6]

$$\|Q(t)\| \leq N_Q \exp(-\sigma_Q t), \;\; \sigma_Q > 0, \;\; t \geq 0;$$

- σ-perturbations [4] (with fixed $\sigma > 0$)

$$\|Q(t)\| \leq N_Q \exp(-\sigma t), \;\; t \geq 0;$$

- power perturbations [1] with arbitrary $\gamma > 0$

$$\|Q(t)\| \leq N_Q t^{-\gamma}, \;\; t \geq 1;$$

- some generalized classes of perturbations [1, 9] similar to previous ones;
- classes defined by various integral conditions [2, 8, 10–13, 16].

Note that everywhere in the above formulas, $N_Q > 0$ is some number depending on Q.

In [1] sharp upper estimates for higher exponent of system (2) with perturbations of the class $\mathfrak{B}[\beta]$ defined by the condition

$$\|Q(t)\| \le N_Q \beta(t), \quad t \ge 0, \tag{3}$$

are obtained when β is some fixed positive piecewise continuous bounded function defined for all $t \ge 0$ and monotone decreasing to 0 with the rate of decrease less than exponential.

Non-monotonic case is partially considered in [11], where β instead of monotonicity obeys the following conditions:

(i) there exists $0<\varepsilon_0<1$ such that for each $\varepsilon\in\,]0,\varepsilon_0[$ the equality $\lim\limits_{m\to\infty} m^{-1} \sum\limits_{k=0}^{m-1} \beta_k^{\varepsilon}=0$ is valid;

(ii) there exists $\rho > 0$ such that for any $k \in \mathbb{N}$ the inequality $\beta_k \le \rho\beta(t)$ holds for each $t \in [k-1, k]$ with the possible exception of a finite number of points.

In these conditions we use the notation $\beta_k = \int\limits_k^{k+1} \beta(t)\,dt$.

It should be stressed that in [1] as well as in [11] the algorithm for evaluation of $\Lambda(\mathfrak{M})$ is similar to the algorithm for evaluation of sigma-exponent due to N. A. Izobov [4].

All the above listed perturbation classes are nondegenerate in the sense that their definitions do not contain any restrictions on the sets $\mathfrak{M}(t) := \{Q(t) : \ Q \in \mathfrak{M}\}$, $t \ge 0$. Indeed, for each of them we have $\mathfrak{M}(t) = \mathbb{R}^{n\times n}$ for all $t \ge 0$. In this report we consider perturbations satisfying the condition (3) with non-negative β. It can be easily seen that $\mathfrak{B}[\beta](t) = 0 \in \mathbb{R}^{n\times n}$ for all t such that $\beta(t) = 0$. Hence, we can assume that $\mathfrak{B}[\beta]$ is to be be considered as one of the simplest examples of perturbation classes with degeneracies. In the future we plan to give a comprehencive consideration of such classes and as a first step in this direction we provide here an estimation of $\Lambda(\mathfrak{B}[\beta])$ for the functions β subject to the natural condition

$$\lim_{m\to\infty} \frac{1}{t} \int\limits_0^t \beta(s)\,ds = 0. \tag{4}$$

We show that N. A. Izobov's algorithm is also applicable in this case.

To obtain the required estimation we use the approach developed in [8, 11–13]. Let $X(t,\tau)$ and $Y(t,\tau)$ be the Cauchy matrices for systems (1) and (2) respectively. Denote $X_k := X(k+1, k)$, $Y_k := Y(k+1,k)$ for $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Take some non-negative piecewise continuous function β defined for all $t \ge 0$ and satisfying condition (4). Put $\beta_k := \int\limits_k^{k+1} \beta(\tau)\,d\tau$, $k \in \mathbb{N}_0$, $b := \sup\limits_{t\ge0} \beta(t)$. Obviously, $b \ge 0$ and $\beta_k \le b$ for all $k \in \mathbb{N}_0$. Now choose arbitrary perturbation $Q \in \mathfrak{B}[\beta]$ satisfying the inequality $\|Q(t)\| \le N_Q\beta(t)$ for all $t \ge 0$ with some $N_Q > 0$.

Lemma 1. *For each $k \in \mathbb{N}_0$ the matrix Y_k can be represented in the form $Y_k = X_k(E + V_k)$ where $V_k \in \mathbb{R}^{n\times n}$ is such that $\|V_k\| \le M\beta_k \le Mb$ and $M := N_Q e^{2a+N_Q b}$.*

Note that unlike [8, 11–13], there is an opportunity for some V_k to be zero for any perturbations $Q \in \mathfrak{B}[\beta]$. Indeed, we have $V_k = 0$ for each $k \notin \mathbb{N}_0^{\beta} := \{k \in \mathbb{N}_0 : \ \beta_k \ne 0\}$.

Denote $\langle m\rangle = \{0, 1, \dots, m-1\}$ for $m \in \mathbb{N}$. Let d be any subset of $\langle m\rangle$. Further we assume that for $d \ne \varnothing$ the elements of d are arranged in the increasing order, so that $d_1 < d_2 < \dots < d_{|d|} =: H(d)$, where $|d|$ is the number of elements of the set d. Thus, $d = \{d_1, d_2, \dots, H(d)\}$.

Define the multipliers V_k, $k \in \mathbb{N}_0$, corresponding to the given perturbation Q by Lemma 1. Consider matrices $S_d^m := \prod\limits_{k=0}^{m-1} X_k W_k(d)$, $m \in \mathbb{N}$, where $W_k(d) = V_k$ if $k \in d$ and $W_k(d) = E$ if $i \notin d$.

Hereinafter we suppose that $\prod$ denotes the product of the factors arranged in descending order of indices. Since $X_{k+s}\cdots X_{k+1}X_k = X(k+s+1,k)$ for any $k,s\in\mathbb{N}_0$, multiplying all X_k with no intermediate multipliers V_k we get

$$S_d^m = X(m,H(d))V_{H(d)}\cdots X(d_2,d_1)V_{d_1}X(d_1,0).$$

Unlike [8, 11–13], here some V_k can be zero and, therefore, S_d^m is nonzero only when $d\subset\mathbb{N}_0^V := \{k\in\mathbb{N}_0:\ V_k\neq 0\}$. Nevertheless, the inequality

$$\|S_d^m\|\le\|X(m,H(d))\|\,\|V_{H(d)}\|\cdots\|X(d_2,d_1)\|\,\|V_{d_1}\|\,\|X(d_1,0)\| =: Z_d(m)$$

remains valid. Since

$$Y(m,0)=\prod_{i=0}^{m-1}X_i(E+V_i)=\sum_{d\subset\langle m\rangle}S_d^m,$$

we can estimate the value of $\|Y(m,0)\|$ by means of $Z_d(m)$.

Theorem 1. *Let h_i, $i\in\mathbb{N}_0$, be a sequence of non-negative numbers such that $h_i>0$ for $i\in\mathbb{N}_0^V$. Then the Cauchy matrix Y of system* (2) *satisfies the inequality*

$$\|Y(m,0)\|\le e^{K(m)}\max_{d\subset\langle m\rangle}R(d)Z_d(m),\ \ m\in\mathbb{N},$$

where $R(d)=\prod\limits_{i\in d}h_i$, $K(m)=\sum\limits_{i\in\langle m,V\rangle}h_i^{-1}$, $\langle m,V\rangle:=\langle m\rangle\cap\mathbb{N}_0^V$.

The following Lemma is a necessary tool to remove condition (i) posed on β in [11].

Lemma 2. *If a sequence of non-negative numbers u_k, $k\in\mathbb{N}_0$, satisfies the condition*

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m-1}u_k=0, \tag{5}$$

then for any $\varepsilon\in\,]0,1]$ the sequence u_k^ε, $k\in\mathbb{N}_0$ satisfies condition (5) *too.*

As in [12], put

$$\Gamma_d^{\varphi}(m)=\|X(m,H(d))\|\varphi(H(d))\cdots\|X(d_2,d_1)\|\,\varphi(d_1)\,\|X(d_1,0)\|,$$

where $\varphi:\mathbb{N}_0\to[0,+\infty[$, $d\subset\langle m\rangle$, $m\in\mathbb{N}$. The main result of our work is given by the following statement.

Theorem 2. *The inequality*

$$\Lambda(\mathfrak{B}[\beta])\le\varlimsup_{m\to\infty}m^{-1}\ln\max_{d\subset\langle m,\beta\rangle}\Gamma_d^{\beta}(m) \tag{6}$$

holds for any non-negative piecewise continuous function β defined for all $t\ge 0$ and satisfying condition (4).

Attainability of the above estimation (6) is a separate problem to be solved by a special version of Millionshchikov's rotation method [14].

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