A Theorem on Differential Inequalities for Linear Functional Differential Equations in Abstract Spaces

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On the interval $[a,b]$, we consider the functional differential equation

$$\boxed{u'(t)=\ell(u)(t)+q(t)} \tag{1}$$

in a Banach space $(\mathbb{X},\|\cdot\|_{\mathbb{X}})$, where $\ell\colon \mathcal{C}([a,b];\mathbb{X})\to\mathcal{B}([a,b];\mathbb{X})$ is a **linear** operator and $q\in\mathcal{B}([a,b];\mathbb{X})$. Here, $\mathcal{C}([a,b];\mathbb{X})$, resp. $\mathcal{B}([a,b];\mathbb{X})$, denotes the Banach space of continuous, resp. Bochner integrable, abstract functions $f\colon [a,b]\to\mathbb{X}$ endowed with the standard norm.

Definition 1. By a solution of equation (1) we understand an abstract function $u\colon [a,b]\to\mathbb{X}$ which is strongly absolutely continuous on $[a,b]$, differentiable a.e. on $[a,b]$, and satisfies equality (1) a.e. on $[a,b]$.

Remark 2. Recall notions of strong absolute continuity and differentiability of abstract functions:

A function $u\colon [a,b]\to\mathbb{X}$ is said to be strongly absolutely continuous, if for each $\varepsilon>0$ there exists $\delta>0$ such that $\sum\limits_i \|u(b_i)-u(a_i)\|_{\mathbb{X}}<\varepsilon$ whenever $\{[a_i,b_i]\}$ is a finite system of mutually non-overlapping subintervals of $[a,b]$ that satisfies $\sum\limits_i (b_i-a_i)<\delta$.

We say that a function $u\colon [a,b]\to\mathbb{X}$ is differentiable at the point $t\in[a,b]$, if there is $\chi\in\mathbb{X}$ such that

$$\lim_{\delta\to0}\Big\|\frac{u(t+\delta)-u(t)}{\delta}-\chi\Big\|_{\mathbb{X}}=0.$$

We denote $\chi=u'(t)$ the derivative of u at t. If u is differentiable at every point $t\in E\subseteq[a,b]$ with $\operatorname{meas}E=b-a$ (in the sense of Lebesgue measure), then u is called differentiable almost everywhere (a.e.) on $[a,b]$.

Remark 3. Differentiability a.e. on $[a,b]$ has to be assumed in Definition 1, because it, generally speaking, does not follow from the strong absolute continuity. Indeed, let $\mathbb{X}=\mathcal{L}([0,1];\mathbb{R})$ and

$$u(t)(x)=\begin{cases}1 & \text{if } 0\le x\le t\le 1,\\ 0 & \text{if } 0\le t<x\le 1.\end{cases}$$

Then u is strongly absolutely continuous on $[0,1]$, but not differentiable a.e. on $[0,1]$ (see [5, Example 7.3.9]).

In what follows, we assume that the Banach space $\mathbb{X}$ is equipped with the **preordering** $\le_K$ generated by a certain **wedge** K. It means that the elements $x_1,x_2\in\mathbb{X}$, by definition, satisfy the relation $x_1\le_K x_2$ if and only if $x_2-x_1\in K$ (we also write $x_2\ge_K x_1$). Recall that, by a wedge (see, e.g., [2]), a closed set $K\subseteq\mathbb{X}$ is understood such that $\alpha_1x_1+\alpha_2x_2\in K$ for arbitrary $\alpha_1,\alpha_2\in[0,+\infty[$ and $x_1,x_2\in K$. It should be noted that the fulfilment of both the relations $x_1\le_K x_2$ and $x_1\ge_K x_2$, generally speaking, does not imply that $x_1=x_2$.

The preordering $\leq_K$ in $\mathbb{X}$ allows one to define a preordering in the space $\mathcal{C}([a,b];\mathbb{X})$ in the following natural way. We say that for abstract functions $f_1, f_2 \in \mathcal{C}([a,b];\mathbb{X})$, the relation $f_1 \leqslant f_2$ holds if $f_1(t) \leq_K f_2(t)$ for every $t \in [a,b]$. However, in order to formulate a main result of this contribution (namely, Theorem 10), we need to introduce a certain strict type inequality in the space $\mathcal{C}([a,b];\mathbb{X})$.

Definition 4. We say that an element $f \in \mathcal{C}([a,b];\mathbb{X})$ is positive and we write $f \blacktriangleright 0$, if for any abstract function $g \in \mathcal{C}([a,b];\mathbb{X})$ there exists a number $\varepsilon > 0$ such that $\varepsilon g \leqslant f$, i.e.,

$$\varepsilon g(t) \leq_K f(t) \text{ for } t \in [a,b].$$

Remark 5. It is easy to see that, in the case $\mathbb{X} = \mathbb{R}$ and $K = [0, +\infty[$, the function $f \in \mathcal{C}([a,b];\mathbb{R})$ satisfies $f \blacktriangleright 0$ if and only if $f(t) > 0$ for $t \in [a,b]$.

Moreover, we assume in Theorem 10 that the operator ℓ in (1) is $\mathcal{B}$-positive in the sense of the following definition.

Definition 6. We say that a linear operator $\ell\colon \mathcal{C}([a,b];\mathbb{X}) \to \mathcal{B}([a,b];\mathbb{X})$ is $\mathcal{B}$-positive if the relation

$$\int_a^t \ell(u)(s)\,\mathrm{d}s \geq_K 0 \text{ for } t \in [a,b]$$

holds for every $u \in \mathcal{C}([a,b];\mathbb{X})$ satisfying

$$u(t) \geq_K 0 \text{ for } t \in [a,b]. \tag{2}$$

Remark 7. It follows from [5, Proposition 5.1.2, Definition 3.2.1] (see also [4, Theorem 4.6]) that for any $g \in \mathcal{B}([a,b];\mathbb{X})$, the implication

$$g(t) \geq_K 0 \text{ for a.e. } t \in [a,b] \implies \int_a^b g(s)\mathrm{d}s \geq_K 0$$

is true. Therefore, a linear operator $\ell\colon \mathcal{C}([a,b];\mathbb{X}) \to \mathcal{B}([a,b];\mathbb{X})$ is $\mathcal{B}$-positive provided that it is positive (increasing), i.e., the relation

$$\ell(u)(t) \geq_K 0 \text{ for a.e. } t \in [a,b]$$

holds for every $u \in \mathcal{C}([a,b];\mathbb{X})$ satisfying (2).

The problem whether the positivity of ℓ is also necessary for its $\mathcal{B}$-positivity is an open question for us.

It is well known that theorems on differential inequalities (maximum principles in other terminology) are powerful tool in the theory of both ordinary and partial differential equations. For abstract differential equation (1), one of possible theorems on differential inequalities can be formulated as follows.

Definition 8. We say that a theorem on differential inequalities holds for equation (1) if the implication

$$\left.\begin{aligned} &u\colon [a,b] \to \mathbb{X} \text{ is strongly absolutely continuous,}\\ &u \text{ is differentiable a.e. on } [a,b], b\\ &u'(t) \geq_K \ell(u)(t) \text{ for a.e. } t \in [a,b],\\ &u(a) \geq_K 0 \end{aligned}\right\} \implies u(t) \geq_K 0 \text{ for } t \in [a,b] \tag{3}$$

is true.

Remark 9. The theorem on differential inequalities formulated in the form implication (3) is connected with the question on the existence, uniqueness, and "sign" of a solution to the Cauchy problem for equation (1), i.e., to the problem

$$u'(t) = \ell(u)(t) + q(t), \quad u(a) = c, \tag{4}$$

where ℓ, q are as in (1) and $c \in \mathbb{X}$.

Assume that $\mathbb{X} = \mathbb{R}$, $K = [0, +\infty[$, and that the theorem on differential inequalities holds for equation (1). Then the homogeneous problem

$$u'(t) = \ell(u)(t), \quad u(a) = 0 \tag{4_0}$$

has only the trivial solution. Indeed, let u be a solution of problem (4_0). Then, by virtue of (3), the inequality $u(t) \geq 0$ holds for $t \in [a,b]$. However, the function $-u$ is also a solution of problem (4_0) and thus, we get $u(t) \leq 0$ for $t \in [a,b]$. Consequently, we have $u \equiv 0$ because $K \cap (-K) = \{0\}$ in the considered particular case. Therefore, assuming (in addition) continuity of ℓ, we derive from the Fredholm alternative (see [1, Theorem 2.1]) that the Cauchy problem (4) is uniquely solvable for any $q \in \mathcal{B}([a,b];\mathbb{X}) = \mathcal{L}([a,b];\mathbb{R})$ and $c \in \mathbb{R}$ and, moreover, implication (3) yields that the corresponding Cauchy operator is positive.

In the case of general $\mathbb{X}$, the situation is much more complicated and needs a further investigation.

Theorem 10. *Let $\ell\colon \mathcal{C}([a,b];\mathbb{X}) \to \mathcal{B}([a,b];\mathbb{X})$ be a linear $\mathcal{B}$-positive operator and there exist a strongly absolutely continuous function $\gamma\colon [a,b] \to \mathbb{X}$, which is differentiable a.e. on $[a,b]$ and satisfies*

$$\gamma \blacktriangleright 0,$$
$$\gamma'(t) \geq_K \ell(\gamma)(t) \ \text{ for a.e. } \ t \in [a,b].$$

Then the theorem on differential inequalities holds for equation (1).

Remark 11. If $\mathbb{X} = \mathbb{R}^n$, $\sigma_1, \dots, \sigma_n \in \{-1, 1\}$,

$$K = \big\{(x_1, \dots, x_n) \in \mathbb{R}^n : \ \sigma_k x_k \geq 0\big\},$$

and ℓ is positive (increasing) in the sense of Remark 7 then, in view of Remark 5, Theorem 10 coincides with one part of [6, Theorem 3.2]. The necessity of the existence of a function γ for the validity of a theorem on differential inequalities in Theorem 10 is still an open question. It is worth mentioning here that, in the case of general $\mathbb{X}$, the necessity indicated cannot be proved so easily as in [6, Theorem 3.2] because neither $K \cap (-K) = \{0\}$ nor the Fredholm alternative holds (some additional assumptions are needed).

Remark 12. One can show (see [7]) that the hyperbolic partial differential equation

$$\frac{\partial y^2(t,x)}{\partial t\, \partial x} = T(y)(t,x) + f(t,x), \tag{5}$$

where $T\colon \mathcal{C}([a,b] \times [c,d];\mathbb{R}) \to \mathcal{L}([a,b] \times [c,d];\mathbb{R})$ is a linear bounded operator[1] and $f \in \mathcal{L}([a,b] \times [c,d];\mathbb{R})$, can be regarded as a particular case of abstract equation (1) in the space $\mathbb{X} = \mathcal{C}([c,d];\mathbb{R})$. Therefore, from Theorem 10 we can derive a result concerning a theorem on differential inequalities for equation (5), which is in a compliance with [3, Theorem 3.1].

[1] $\mathcal{C}([a,b] \times [c,d];\mathbb{R})$, resp. $\mathcal{L}([a,b] \times [c,d];\mathbb{R})$, denotes the Banach space of continuous, resp. Lebesgue integrable, functions $y\colon [a,b] \times [c,d] \to \mathbb{R}$ endowed with the standard norm.

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