Studying Integro-Differential CNN Model with Applications in Nano-Technology

E. Litsyn

Department of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva, Israel
E-mail: elena.litsyn@weizmann.ac.il

A. Slavova, G. Bobeva

Institute of Mathematics, Bulgarian Academy of Sciences, Sofia 1113, Bulgaria
E-mail: slavova@math.bas.bg; galbobeva@gmail.com

1 Introduction

The demand for smaller and faster devices has encouraged technological advances resulting in the ability to manipulate matter at nanoscales that have enabled the fabrication of nanoscale electromechanical systems. With the advances in materials synthesis and device processing capabilities, the importance of developing and understanding nanoscale engineering devices has dramatically increased over the past decade. For this purpose we shall study integro-differential equations for solution of dynamic coupled problems in multifunctional nano-heterogeneous piezoelectric composites.

Let $G \in R^2$ is a bounded piezoelectric domain with a set of inhomogeneities $I = \cup I_k \in G$ (holes, inclusions, nano-holes, nano-inclusions, which means that their diameter is less than $10^{-7}m$, see Figure 1.

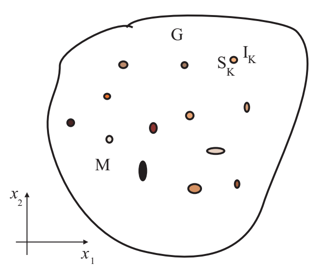

Figure 1. The geometry: PEM inclusions in a bounded PEM matrix.

The aim is to find the field in every point of $M = G \setminus I$, I and to evaluate stress concentration around the inhomogeneities. For this purpose we shall consider the case, when I is a nano-inclusion and we shall assume the following boundary conditions on S:

$$t_p^M = \frac{\partial \sigma_{lp}^S}{\partial l} \text{ on } S, \text{ or } \tau_3^I + t_3^M = \frac{\partial \sigma_{l3}^S}{\partial l}, \quad \tau_4^I + t_4^M = \frac{\partial \sigma_{l4}^S}{\partial l}, \tag{1.1}$$

where σ_{lp}^S is generalized stress [3], $p = 3, 4$, l is the tangential vector. In [3] the above formulated task is reduced to integro-differential equation. In this paper we shall consider CNN integro-differential model of the problem under consideration and we shall study its dynamics. We shall provide computer simulations for the evaluation of dynamic SCF (stress concentration factor). This characteristic is of interest in nano-mechanics and it is denoted by $|\sigma_{\varphi 3}/\sigma_0|$. Another characteristics of importance in nano-technology is the normalized dynamic Electric Field Concentration Field (EFCF) $|e_{15}^M E_\varphi/\sigma_0|$ along the perimeter of the inhomogeneity. Here φ is the polar angle of the observer point.

2 Dynamics of CNN integro-differential model

In [3] a system of integro-differential equations (IDE) is obtained for the unknowns u (displacement vectors) and τ (traction). The procedure is based on Gauss theorem [7] after finding the fundamental solutions of the boundary value problem formulated in the introduction.

In this section we shall consider the following system of integro-differential equations which is more general from the point of view of the applications in nano-technology:

$$u_t - u_{xx} = F(u, \tau) - b \int_0^t u(s, x)\, ds, \tag{2.1}$$

$u(x,t)$, $0 < x, t < 1$, $b = const$. The proposed system (2.1) is a system of nonlinear integro-differential equation, in which $F(u)$ is a function of displacement vectors and the traction (u, τ) [3].

We shall construct CNN architecture of the above IDE (2.1). First, we map $u(x,t)$ into a CNN layer such that the state voltage of a CNN cell $v_{xij}(t)$ at a grid point (i,j) is associated with $u(ih, jh, t)$, $h = \Delta x$ and using the two-dimensional discretized Laplacian template $A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, it is easy to design the CNN model:

(1) CNN cell dynamics:

$$\frac{du_{ij}}{dt} - I_{ij}^s = F(u_{ij}, \tau) - b \int_0^t u_{ij}(s)\, ds. \tag{2.2}$$

(2) CNN synaptic law:

$$I_{ij}^s = \frac{1}{h^2}\,(u_{ij-1} + u_{ij+1} - 4u_{ij} + u_{i-1j} + u_{i+1j}). \tag{2.3}$$

Let us assume for simplicity that the grid size of our CNN model is $h = 1$. Substituting (2.3) into (2.2) we obtain:

$$\frac{du_{ij}}{dt} - A_2 * u_{ij} = F(u_{ij}) - b \int_0^t u_{ij}(s)\, ds, \ \ 1 \le i, j \le N. \tag{2.4}$$

The obtained CNN model (2.4) is actually a system of IDE which is identified as the state equation of an autonomous CNN made of $N \times N$ cells [1, 5].

We shall study the dynamics of the CNN integro-differential model (2.4) by means of the theory of local activity [2, 4]. The theory which will be presented below offers a constructive analytical method for uncovering local activity. One can determine the domain of the cell parameters in order for the cells to be locally active, and thus potentially capable of exhibiting complexity. This precisely defined parameter domain is called the edge of chaos.

Following the theory of local activity we shall find the equilibrium points E of (2.4) [6]. In general, the equilibrium points are functions of the cell parameters. We shall consider the equilibrium point $E^0 = (0, 0)$. We calculate the four cell coefficients a_{11}, a_{12}, a_{21}, a_{22} of the Jacobian matrix at equilibrium point $E^0 = (0, 0)$ and as well the trace $\mathrm{Tr}(E^0)$ and determinant $\Delta(E^0)$.Then we define stable and locally active region for the CNN integro-differential model (2.4).

Definition 2.1. We say that the cell is both stable and locally active at the equilibrium point E^0 for the CNN integro-differential model (2.4) if

$$a_{22} > 0 \ \text{ or } \ 4a_{11}a_{22} < (a_{12} + a_{21})^2$$

and

$$\mathrm{Tr}(E^0) < 0, \quad \Delta(E^0) > 0.$$

This region in the parameter space is called $\mathrm{SLAR}(E^0)$.

According to [2,4] edge of chaos (EC) is a region in the parameter space of a dynamical system in which emergence of complex phenomena and information processing is possible. Until now the definition of this phenomena is known only via empirical examples. Below we give more precise mathematical definition of EC.

Definition 2.2. CNN integro-differential model (2.4) operates in edge of chaos regime if and only if there is least one equilibrium point such that the cell is both locally active and stable.

Then the following theorem holds:

Theorem. *CNN integro-differential model* (2.4) *operates in edge of chaos if and only if the following conditions are satisfied:* $-1 < b < 1$, $F(0) = 0$, $F < 0 \in (0, b)$, $F > 0 \in (b, 1)$, $F'(0) < 0$, $F'(1) < 0$. *This means that there is at least one equilibrium point which is both locally active and stable.*

Remark. It is very important to have circuit model for the physical implementation. Then we can apply results from the classical circuit theory in order to justify the cells local activity. If the cell acts like a source of small signal for at least one equilibrium point then we can say that it is locally active. In this case the cell can inject a net small-signal average power into the passive resistive grids [2,4].

3 Simulations

In this section we shall consider an illustrative example. Let us consider the domain $G_1G_2G_3G_4$ in Figure 2, which is a square elastic isotropic plate under uniform uni-axial time-harmonic traction of magnitude σ_0 applied to the vertical boundaries.

The heterogeneity is presented by a circular nano-inclusion with radius a. The size of the square plate is $10d$, where $d = 2a$. A dimensionless parameter is introduced and it is defined as $s = C_S/2\mu^M a$, where μ^M is the shear modulus of the plate material, $C_S = \lambda^S + 2\mu^S$. When the heterogeneity is presented by the inclusion the stiffness ratio of both phases is $\mu^I/\mu^M = 0.2$ and the densities correspond to frequency ratio of $\Omega^I/\Omega^M = 3.0$, where $\Omega^J = \omega a\sqrt{\rho^J/\mu^J}$, $J = I, M$. In all simulations the material damping is set to 5% and Poisson's ratio is 0.26 for both matrix and inclusion. The normalized hoop stresses spectra for representative point with polar angle $\phi = \pi/2$ of the heterogeneity interface versus normalized frequency for a single hole and inclusion cases are plotted in Figure 3. The dynamic SCF is defined as $|\sigma_{\phi\phi}/\sigma_0|$. Four different values of the surface parameter are considered namely $s = 0; 0.1; 0.5; 1.0$. The problem is solved for frequency range up to $\Omega^M = 0.8$.

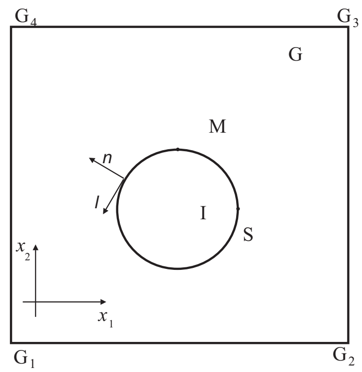

Figure 2. Rectangular PEM matrix with circle heterogeneity.

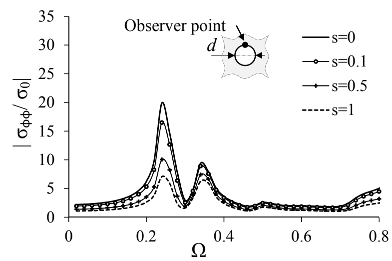

Figure 3. SCF versus frequency at observer point $\phi = \pi/2$ along interface between finite elastic isotropic matrix for nano-inclusion.

4 Conclusion

Time-harmonic elastodynamic analysis of anisotropic finite solids with defects such as nano-sized inclusions is presented in this work. The mathematical model combines classical 2D elastodynamic theory and surface elasticity model [3] allowing in such way to treat heterogeneities at nano-level. The analysis is carried out on IDE that employed the appropriate frequency-dependent fundamental solution, obtained with Radon transform [7]. The CNN architecture is implemented numerically by discretization of the IDE under consideration (2.1). Finally, numerical simulations show that the stress concentration field near defects is strongly influenced by the type and the size of the inclusion, the material anisotropy, the defect location and geometry, the dynamic load characteristics and the mutual interactions between defects and between them and the solid's boundary. The results of the present methodology are with application in the fields of computational fracture mechanics, geotechnical engineering and non-destructive testing evaluation of anisotropic composite materials.

Acknowledgement

E. Litsyn and A. Slavova – this paper is performed in the frames of working program on the Bilateral Res. Project between Bulgarian Academy of Sciences and Israel Academy of Sciences.

G. Bobeva – this paper is supported by the project DFNP-17-26.

References

[1] L. O. Chua and L. Yang, Cellular neural networks: applications. *IEEE Transactions on Circuits and Systems* **35** (1988), no. 10, 1273–1290.

[2] L. O. Chua, Local activity is the origin of complexity. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **15** (2005), no. 11, 3435–3456.

[3] P. Dineva, D. Gross, R. Müller and T. Rangelov, *Dynamic Fracture of Piezoelectric Materials. Solution of Time-Harmonic Problems Via BIEM.* Solid Mechanics and its Applications, 212. Springer, Cham, 2014.

[4] K. Mainzer and L. Chua, *Local Activity Principle.* Imperial College Press, London, 2013.

[5] A. Slavova, *Cellular Neural Networks: Dynamics and Modeling*, Kluwer Academic Publishers, Dordrecht–Boston–London, 2003.

[6] M. Vidyasagar, *Nonlinear Systems Analysis.* Reprint of the second (1993) edition. Classics in Applied Mathematics, 42. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002.

[7] V. S. Vladimirov, *Equations of Mathematical Physics.* Translated from the Russian by Audrey Littlewood. Edited by Alan Jeffrey. Pure and Applied Mathematics, 3 Marcel Dekker, Inc., New York, 1971.