On the Existence of Some Solutions of Systems of Ordinary Differential Equations which is Partially Resolved Relatively to the Derivatives in the Case of Fixed Singularity

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Let us consider the system of ordinary differential equations:

$$A(z)Y' = B(z)Y + f(z, Y, Y'), \tag{0.1}$$

where matrices $A, B : D_1 \to \mathbb{C}^{p\times n}$, $D_1 = \{z : \ |z| < R_1, \ R_1 > 0\} \subset \mathbb{C}$, matrices $A(z)$, $B(z)$ are analytic in the domain D_{10}, $D_{10} = D_1 \setminus \{0\}$, the pencil of matrices $A(z)\lambda - B(z)$ is singular on the condition that $z \to 0$, function $f : D_1 \times G_1 \times G_2 \to \mathbb{C}^p$, where domains $G_k \subset \mathbb{C}^n$, $0 \in G_k$, $k = 1, 2$, function $f(z, Y, Y')$ is analytic in $D_{10} \times G_{10} \times G_{20}$, $G_{k0} = G_k \setminus \{0\}$, $k = 1, 2$.

The system of ordinary differential equations (0.1) that satisfies conditions $p < n$, $A(z)$ is analytic matrix in the domain D_1 and $\operatorname{rang} A(z) = p$ on condition that $z \in D_1$.

Let us consider the function

$$Y = \operatorname{col}\begin{pmatrix} Y_1 & Y_2 \end{pmatrix}, \quad Y_1 : D_1 \to \mathbb{C}^p, \quad Y_2 : D_1 \to \mathbb{C}^{n-p}, \quad Y_1 = \operatorname{col}\big(Y_{11}(z), \dots, Y_{1p}(z)\big),$$
$$Y_2 = \operatorname{col}\big(Y_{21}(z), \dots, Y_{2n-p})(z)\big).$$

Without restricting the generality, assume that matrices $A(z)$, $B(z)$ and vector-function $f(z, Y, Y')$ take the forms:

$$A(z) = \begin{pmatrix} A_1(z) & A_2(z) \end{pmatrix}, \quad B(z) = \begin{pmatrix} B_1(z) & B_2(z) \end{pmatrix}, \quad f(z, Y, Y') = f^*(z, Y_1, Y_2, Y_1', Y_2'),$$

$A_1 : D_1 \to \mathbb{C}^{p\times p}$, $A_2 : D_1 \to \mathbb{C}^{p\times(n-p)}$, $B_1 : D_1 \to \mathbb{C}^{p\times p}$, $B_2 : D_1 \to \mathbb{C}^{p\times(n-p)}$, $\det A_1(z) \neq 0$ on the condition that $z \in D_1$, $f^* : D_1 \times G_{11} \times G_{12} \times G_{21} \times G_{22} \to C^p$, $G_{j1} \times G_{j2} = G_j$, $G_{j1} \subset C^p$, $G_{j2} \subset C^{n-p}$, $j = 1, 2$.

In this view the system (0.1) may be written as:

$$Y_1' = A_1^{-1}(z)B_1(z)Y_1 + A_1^{-1}(z)B_2(z)Y_2 - A_1^{-1}(z)A_2(z)Y_2' + A_1^{-1}(z)f^*(z, Y_1, Y_2, Y_1', Y_2'). \tag{0.2}$$

Let us suppose that matrices $A_1^{-1}(z)B_1(z)$, $A_1^{-1}(z)A_2(z)$, $A_1^{-1}(z)B_2(z)$ are analytic in the domain D_{10} and have removable singularity in the point $z = 0$.

Let us introduce the following notation:

$$P(z) = A_1^{-1}(z)B_1(z),$$
$$F^*(z, Y_1, Y_2, Y_1', Y_2') = A_1^{-1}(z)B_2(z)Y_2 - A_1^{-1}(z)A_2(z)Y_2' + A_1^{-1}f^*(z, Y_1, Y_2, Y_1', Y_2'),$$

then the system (0.2) may be written as

$$Y_1' = P(z)Y_1 + F^*(z, Y_1, Y_2, Y_1', Y_2'), \tag{0.3}$$

where $P(z)$ is analytic matrix in the domain D_{10} and has removable singularity in the point $z = 0$, $F^*(z, Y_1, Y_2, Y_1', Y_2')$ is analytic vector-function in the domain $D_{10} \times G_{110} \times G_{120} \times G_{210} \times G_{220}$, $G_{jk0} = G_{jk} \setminus 0$, $j, k = 1, 2$.

Let us introduce the following classes of functions:

- By H_0^{n-p} we basically mean class of $(n-p)$-dimensional analytic in the domain D_{10} functions that have removable singularity in the point $z=0$.
- By H_r^{n-p} we basically mean class of $(n-p)$-dimensional analytic in the domain D_{10} functions that have pole of r-order in the point $z=0$.

We study the system (0.3) that satisfies the hypothesis that $Y_2(z)$ is arbitrary state function from given class of function.

Let us consider the following two cases:

- vector-function Y_2 appertain to class of functions H_0^{n-p},
- vector-function Y_2 appertain to class of functions H_r^{n-p}.

1 Case when the function Y_2 has removable singularity at the point $z=0$

In the case $Y_2 \in H_0^{n-p}$, let us study question on the existence of the analytic solutions of Cauchy's problem

$$\begin{cases} Y_1' = P(z)Y_1 + F^*(z, Y_1, Y_2, Y_1', Y_2'), \\ Y_1(z) \to 0 \ \text{ on the condition that } \ z \to 0, \ \ z \in D_{10}, \end{cases} \tag{1.1}$$

that satisfies the additional condition

$$Y_1'(z) \to 0 \ \text{ on the condition that } \ z \to 0, \ \ z \in D_{10}. \tag{1.2}$$

Let us choose such vector-function $Y_2 \in H_0^{n-p}$ that after regularization in the point $z=0$, becomes analytic function in the domain D_1 and $Y_2(0)=0$.

In this case, the function F^* may be written as

$$F^*(z, Y_1, Y_2, Y_1', Y_2') = F^*\Big(z, Y_1, \sum_{k=1}^{\infty} A_k z^k, Y_1', \sum_{k=1}^{\infty} k \cdot A_k z^{k-1}\Big) = F(z, Y_1, Y_1'),$$

where $F : D_1 \times G_{11} \times G_{21} \to \mathbb{C}^p$.

Thus the problem (1.1) could be reduce to Cauchy's problem:

$$\begin{cases} Y_1' = P(z)Y_1 + F(z, Y_1, Y_1'), \\ Y_1(z) \to 0 \ \text{ on the condition that } \ z \to 0, z \in D_{10}. \end{cases} \tag{1.3}$$

The sufficient conditions were found in which for each arbitrary fixed function $Y_2 \in H_0^{n-p}$, $Y_2(0)=0$, there exists at least one analytic solution of Cauchy's problem (1.3) with the additional condition (1.2) in some subdomain of the domain D_{10} with point $z=0$ at the domain boundary.

2 Case when the function Y_2 has the pole of r-order at the point $z=0$

In this case, let us study question on existence of the analytic solutions of Cauchy's problem (1.1) satisfying the additional condition (1.2) for each arbitrary fixed function $Y_2 \in H_r^{n-p}$.

By condition, the function $Y_2 \in H_r^{n-p}$ may be written as

$$Y_2(z) = z^{-r} Y_2^*(z),$$

where $Y_2^*(z)$ is a analytic function in the domain D_1, and $Y_2^*(0) \neq 0$, moreover, function $Y_2^*(z)$ may be submitted in convergent power series on the condition that $z \in D_1$.

Let us suppose that the power series expansion of function F^* in the domain of point $(0,0,0,0,0)$ has finite number of summand containing vector-functions Y_2 and Y_2'.

Then vector-function $F^*(z, Y_1, Y_2, Y_1', Y_2')$ may be written as

$$F^*(z, Y_1, Y_2, Y_1', Y_2') = z^{-l} \cdot F(z, Y_1, Y_2^*, Y_1', Y_2^{*\prime}),$$

where vector-function $F(z, Y_1, Y_2^*, Y_1', Y_2^{*\prime})$ is analytic function in the domain $D_1 \times G_{11} \times G_{12} \times G_{21} \times G_{22}$, $l \in \mathbb{N}$, $l \geq r+1$.

The system (0.3) may be written as

$$\begin{aligned} z^l Y_1' = z^l A_1^{-1}(z) B_1(z) Y_1 - z^{l-r-1} A_1^{-1}(z) A_2(z) Y_2^{*\prime} \\ + z^{l-r} A_1^{-1}(z) B_2(z) Y_2^* + F(z, Y_1, Y_2^*, Y_1', Y_2^{*\prime}). \end{aligned} \tag{2.1}$$

Let us introduce the following notation

$$P(z) = A_1^{-1}(z) B_1(z), \quad R(z) = A_1^{-1}(z) A_2(z), \quad C(z) = A_1^{-1}(z) B_2(z).$$

Then the system (2.1) may be written

$$z^l Y_1' = z^l P(z) Y_1 - z^{l-r-1} R(z) Y_2^{*\prime} + z^{l-r} C(z) Y_2^* + F(z, Y_1, Y_2^*, Y_1', Y_2^{*\prime}), \tag{2.2}$$

where $P(z)$, $R(z)$, $C(z)$ are analytic matrices in the domain D_1.

The questions on the analytic solutions of Cauchy's problem existence (2.2) that satisfy the initial condition

$$Y_1(z) \to 0 \ \text{ on the condition that } \ z \to 0, \quad z \in D_{10}, \tag{2.3}$$

and the additional condition:

$$Y_1'(z) \to 0 \ \text{ on the condition that } \ z \to 0, \quad z \in D_{10}, \tag{2.4}$$

are considered.

The sufficient conditions were found on which for each arbitrary fixed function $Y_2 \in H_r^{n-p}$, there exists at least one analytic solution of Cauchy's problem (2.2), (2.3) with the additional condition (2.4) in some subdomain of the domain D_{10} with point $z = 0$ at the domain boundary.

For each of these cases we researched the properties of the relevant solutions of the system (0.1).

References

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