# Convergence of Finite Difference Scheme and Uniqueness of a Solution for One System of Nonlinear Integro-Differential Equations with Source Terms

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Mathematical models of diffusive processes lead to nonstationary partial integro-differential equations and systems of those equations. Most of those problems, as a rule, are nonlinear. This moment significantly complicates the investigation of such models.

Our goal is to investigate and study numerical solutions of nonlinear integro-differential diffusion system, which appears at mathematical modeling of process of electromagnetic field propagation into a substance. The main characteristic of the corresponding system of Maxwell's equations is that it contains equations, which are strongly connected to each other. This circumstance dictates to use the corresponding investigation methods for each concrete model, as the general theory even for such linear systems is not yet fully developed. Naturally, the questions of numerical solution of these problems, which also are connected with serious complexities, arise as well.

In particular, our purpose is to study the following system of nonlinear integro-differential equations with source terms:

$$\begin{gathered}
\frac{\partial U}{\partial t}-\frac{\partial}{\partial x}\left[\left(1+\int\limits_0^t\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]d\tau\right)^p\frac{\partial U}{\partial x}\right]+|U|^{q-2}U=0,\\
\frac{\partial V}{\partial t}-\frac{\partial}{\partial x}\left[\left(1+\int\limits_0^t\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]d\tau\right)^p\frac{\partial V}{\partial x}\right]+|V|^{q-2}V=0,
\end{gathered}\tag{1}$$

where $0<p\leq 1$, $q\geq 2$.

System above is obtained by adding the source terms to the resulting model which is derived after reduction of well-known Maxwell's equations to the system of nonlinear integro-differential equations. Such a reduction at first was made in [4].

The system of Maxwell's equation can be done in the following form [12]:

$$\frac{\partial H}{\partial t}=-\operatorname{rot}(\nu_m\operatorname{rot}H),\tag{2}$$

$$c_\nu\,\frac{\partial\theta}{\partial t}=\nu_m(\operatorname{rot}H)^2,\tag{3}$$

where $H=(H_1,H_2,H_3)$ is a vector of the magnetic field, $\theta$ is temperature, $c_\nu$ and $\nu_m$ characterizes the thermal heat capacity and electro-conductivity of the substance, respectively.

While propagating in the medium, variable magnetic field induces a variable electric field that generates a current. The current causes increase the medium's temperature, which should be taken into account for further investigations. Thus, we can say that coefficients $c_\nu=c_\nu(\theta)$ and

$\nu_m = \nu_m(\theta)$ are functions of temperature. After integration of (3) by time and substituting it in (2) the corresponding system of Maxwell's equations can be reduced to the following integro-differential form [4]:

$$\frac{\partial H}{\partial t} = -\operatorname{rot}\bigg[a\bigg(\int\limits_0^t |\operatorname{rot} H|^2\,d\tau\bigg)\operatorname{rot} H\bigg]. \tag{4}$$

The literature on the questions of existence, uniqueness, and regularity of solutions to the equations of the above types is very rich. In [1,2,4] the solvability of the first boundary value problem for scalar cases is studied using a modified version of the Galerkin's method and compactness arguments that are used in [13,14] for investigation of elliptic and parabolic models. The uniqueness of the solutions is investigated also in [1, 2, 4]. The asymptotic behavior of solutions is discussed in [3, 6–9, 11] and in a number of other works as well. Note also that to numerical resolution of (4) type one-dimensional equations were devoted many works as well, see, e.g., [5, 9–11] and references therein.

If we consider two component magnetic field $H = (0, U, V)$, where $U = U(x,t)$ and $V = V(x,t)$, then from (3) we get system of nonlinear integro-differential equations (1) with source terms. In the domain $[0,1]\times[0,\infty)$ for system (1) let us consider the following initial-boundary value problem:

$$U(0,t) = U(1,t) = V(0,t) = V(1,t) = 0, \quad t \geq 0, \tag{5}$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \quad x \in [0,1], \tag{6}$$

where $U_0$ and $V_0$ are given functions.

Let us consider the semi-discrete scheme for problem (1), (5), (6). On $[0,1]$ let us introduce a net with mesh points denoted by $x_i = ih$, $i = 0, 1, \dots, M$, with $h = 1/M$. The boundaries are specified by $i = 0$ and $i = M$. The semi-discrete approximation at $(x_i, t)$ are designed by $u_i = u_i(t)$ and $v_i = v_i(t)$. The exact solution to the problem at $(x_i, t)$ is denoted by $U_i = U_i(t)$ and $V_i = V_i(t)$. At points $i = 1, 2, \dots, M-1$, the integro-differential equation will be replaced by approximation of the space derivatives by a forward and backward differences.

Let us correspond to (1), (5), (6) problem the following semi-discrete scheme:

$$\begin{gathered}
\left.\begin{aligned}
&\frac{du_i}{dt} - \bigg[\bigg(1 + \int\limits_0^t \big[(u_{\overline{x},i})^2 + (v_{\overline{x},i})^2\big]\,d\tau\bigg)^p u_{\overline{x},i}\bigg]_x + |u_i|^{q-2}u_i = 0,\\
&\frac{dv_i}{dt} - \bigg[\bigg(1 + \int\limits_0^t \big[(u_{\overline{x},i})^2 + (v_{\overline{x},i})^2\big]\,d\tau\bigg)^p v_{\overline{x},i}\bigg]_x + |v_i|^{q-2}v_i = 0,
\end{aligned}\right.\quad i = 1, 2, \dots, M-1,\\
u_0(t) = u_M(t) = v_0(t) = v_M(t) = 0,\\
u_i(0) = U_{0,i}, \qquad v_i(0) = V_{0,i}, \quad i = 0, 1, \dots, M,
\end{gathered} \tag{7}$$

where

$$r_x = \frac{r_{i+1} - r_i}{h}, \quad r_{\overline{x}} = \frac{r_i - r_{i-1}}{h}.$$

So, we obtained the Cauchy problem (7) for nonlinear system of ordinary integro-differential equations.

It is not difficult to obtain the following estimates:

$$\|u(t)\|^2 + \int\limits_0^t \|u_{\overline{x}}]|^2\,d\tau \leq C, \quad \|v(t)\|^2 + \int\limits_0^t \|v_{\overline{x}}]|^2\,d\tau \leq C, \tag{8}$$

where
$$\|w(t)\|^2 = \sum_{i=1}^{M-1} w_i^2(t)h, \quad \|w_{\overline{x}}]|^2 = \sum_{i=1}^{M} w_{\overline{x},i}^2(t)h.$$

The a priori estimates (8) guarantee the global solvability of problem (7).

The following statement is true.

**Theorem 1.** *If* $0 < p \le 1$, $q \ge 2$ *and problem* $(1), (5), (6)$ *have a sufficiently smooth solution* $U(x,t)$, $V(x,t)$, *then the solution of problem* $(7)$
$$u = u(t) = (u_1(t), u_2(t), \dots, u_{M-1}(t)), \quad v = v(t) = (v_1(t), v_2(t), \dots, v_{M-1}(t))$$
*tends to*
$$U = U(t) = (U_1(t), U_2(t), \dots, U_{M-1}(t)), \quad V = V(t) = (V_1(t), V_2(t), \dots, V_{M-1}(t))$$
*as* $h \to 0$ *and the following estimates are true:*
$$\|u(t) - U(t)\| \le Ch, \quad \|v(t) - V(t)\| \le Ch.$$

Now let us consider the fully discrete scheme for problem $(1), (5), 6$. On $[0,1] \times [0,T]$ let us introduce a net with mesh points denoted by $(x_i, t_j) = (ih, j\tau)$, where $i = 0, 1, \dots, M$; $j = 0, 1, \dots, N$ with $h = 1/M$, $\tau = T/N$. The initial line is denoted by $j = 0$. The discrete approximation at $(x_i, t_j)$ is designed by $(u_i^j, v_i^j)$ and the exact solution to problem $(1), (5), (6)$ by $(U_i^j, V_i^j)$.

For problem $(1), (5), (6)$ let us consider the following finite difference scheme:
$$\begin{gathered}
\frac{u_i^{j+1} - u_i^j}{\tau} - \bigg\{ \Big( 1 + \tau \sum_{k=1}^{j+1} \big[ (u_{\overline{x},i}^k)^2 + (v_{\overline{x},i}^k)^2 \big] \Big)^p u_{\overline{x},i}^{j+1} \bigg\}_x + |u_i^{j+1}|^{q-2} u_i^{j+1} = f_{1,i}^j, \\
\frac{v_i^{j+1} - v_i^j}{\tau} - \bigg\{ \Big( 1 + \tau \sum_{k=1}^{j+1} \big[ (u_{\overline{x},i}^k)^2 + (v_{\overline{x},i}^k)^2 \big] \Big)^p v_{\overline{x},i}^{j+1} \bigg\}_x + |v_i^{j+1}|^{q-2} v_i^{j+1} = f_{2,i}^j, \\
i = 1, 2, \dots, M-1; \;\; j = 0, 1, \dots, N-1, \\
u_0^j = u_M^j = v_0^j = v_M^j = 0, \;\; j = 0, 1, \dots, N, \\
u_i^0 = U_{0,i}, \quad v_i^0 = V_{0,i}, \;\; i = 0, 1, \dots, M.
\end{gathered} \tag{9}$$

Multiplying equations in (9) scalarly by $u_i^{j+1}$ and $v_i^{j+1}$ respectively, it is not difficult to get the inequalities:
$$\|u^n\|^2 + \sum_{j=1}^{n} \|u_{\overline{x}}^j]|^2 \tau < C, \quad \|v^n\|^2 + \sum_{j=1}^{n} \|v_{\overline{x}}^j]|^2 \tau < C, \;\; n = 1, 2, \dots, N, \tag{10}$$
where here and below $C$ is a positive constant independent from $\tau$ and $h$.

The a priori estimates (10) guarantee the stability of scheme (9). The main statement of this note can be stated as follows.

**Theorem 2.** *If* $0 < p \le 1$, $q \ge 2$ *and problem* $(1), (5), (6)$ *has a sufficiently smooth solution* $(U(x,t), V(x,t))$, *then the solution* $u^j = (u_1^j, u_2^j, \dots, u_M^j)$, $v^j = (v_1^j, v_2^j, \dots, v_M^j)$, $j = 1, 2, \dots, N$ *of the difference scheme* $(9)$ *tends to the solution of the continuous problem* $(1), (5), (6)$ $U^j = (U_1^j, U_2^j, \dots, U_M^j)$, $V^j = (V_1^j, V_2^j, \dots, V_M^j)$, $j = 1, 2, \dots, N$ *as* $\tau \to 0$, $h \to 0$ *and the following estimates are true:*
$$\|u^j - U^j\| \le C(\tau + h), \quad \|v^j - V^j\| \le C(\tau + h).$$

We have carried out numerous numerical experiments for problem $(1), (5), (6)$ with different kind of right hand sides and initial-boundary conditions.

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