On the Cauchy Weighted Problem for Higher Order Singular Ordinary Differential Equations

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On a finite semi-open interval $]a,b]$ we consider the differential equation

$$u^{(n)} = f(t, u, \dots, u^{(n-1)}) \tag{1}$$

with the weighted initial conditions

$$\lim_{t\to a} u^{(i-1)}(t) = 0 \ \ (i = 1, \dots, n-1), \quad \limsup_{t\to a} \frac{|u^{(n-1)}(t)|}{\delta(t)} < +\infty, \tag{2}$$

where $n \geq 2$, the functions $f :]a,b] \times \mathbb{R}^n \to \mathbb{R}$ and $\delta :]a,b] \to]0,+\infty[$ are continuous and

$$\int_a^b \delta(t)\,dt < +\infty.$$

Equation (1) is said to be singular in the time variable if

$$\int_a^b f^*(t,x)\,dt = +\infty \ \text{ for } \ x > 0,$$

where

$$f^*(t,x) = \max\Big\{|f(t,x_1,\dots,x_n)| : \ \sum_{i=1}^n |x_i| \leq x\Big\}.$$

For such equations, the weighted initial problem has been investigated earlier only in the case, when

$$\int_a^b |f(t,0,\dots,0)|\,dt < +\infty, \quad \lim_{t\to a} \delta(t) = 0.$$

We have established unimprovable in a certain sense sufficient conditions for solvability and unique solvability of problem $(1),(2)$ covering the cases, where

$$\int_a^b (t-a)^\mu |f(t,0,\dots,0)|\,dt = +\infty \ \text{ for any } \ \mu > 0,$$

or the weighted function δ has no finite limit at the point a.

In Theorems 1–3 formulated below and dealing with the solvability, unsolvability, and unique solvability of problem $(1),(2)$, respectively, it is assumed that the function f on the set $]a,b] \times \mathbb{R}^n$ satisfies one of the following three conditions:

$$f(t,x_1,\dots,x_n)\operatorname{sgn}(x_n) \leq \sum_{i=1}^n h_i(t)|x_i| + h_0(t), \tag{3}$$

$$f(t,x_1,\dots,x_n)-h_n(t)x_n \geq \sum_{i=1}^{n-1} h_i(t)|x_i| + h_0(t), \tag{4}$$

$$\big[f(t,x_1,\dots,x_n)-f(t,y_1,\dots,y_n)\big]\operatorname{sgn}(x_n-y_n) \leq \sum_{i=1}^{n} h_i(t)|x_i-y_i|, \tag{5}$$

where $h_i :]a,b] \to [0,+\infty[$ $(i=0,\dots,n-1)$ and $h_n :]a,b] \to \mathbb{R}$ are continuous functions. Note that unlike h_i $(i=0,\dots,n-1)$, the function h_n may be negative or with alternating sign.

Suppose

$$\delta_i(t) = \frac{1}{(n-1-i)!}\int_a^t (t-s)^{n-1-i}\delta(s)\,ds \;\; (i=1,\dots,n-1).$$

Theorem 1. *If along with* (3) *the inequalities*

$$\limsup_{t\to a}\bigg[\frac{1}{\delta(t)}\sum_{i=1}^{n-1}\int_a^t \exp\bigg(\int_s^t h_n(\tau)\,d\tau\bigg)\delta_i(s)h_i(s)\,ds\bigg] < 1, \tag{6}$$

$$\limsup_{t\to a}\bigg[\frac{1}{\delta(t)}\int_a^t \exp\bigg(\int_s^t h_n(\tau)\,d\tau\bigg)h_0(s)\,ds\bigg] < +\infty \tag{7}$$

are fulfilled, then problem (1), (2) *has at least one solution.*

Theorem 2. *Let along with* (4) *the conditions*

$$\liminf_{t\to a}\bigg[\delta(t)\exp\bigg(\int_t^b h_n(s)\,ds\bigg)\bigg] = 0, \tag{8}$$

$$\liminf_{t\to a}\bigg[\frac{1}{\delta(t)}\int_a^t \exp\bigg(\int_s^t h_n(\tau)\,d\tau\bigg)h_0(s)\,ds\bigg] > 0$$

be fulfilled and there exist $b_0 \in]a,b[$ *such that*

$$\sum_{i=1}^{n-1}\int_a^t \exp\bigg(\int_s^t h_n(\tau)\,d\tau\bigg)\delta_i(s)h_i(s)\,ds \geq \delta(t) \;\; \text{for} \;\; a<t\leq b_0.$$

Then problem (1), (2) *has no solution.*

Assume now that the function δ is continuously differentiable on $]a,b]$ and, as an example, consider the differential equation

$$u^{(n)} = h_n(t)u^{(n-1)} + \sum_{i=1}^{n} h_i(t)|u^{(i-1)}| + f_0(t,u,\dots,u^{(n-1)}), \tag{9}$$

where

$$h_n(t) = \frac{\delta'(t)}{\delta(t)} - h(t), \;\; h_i(t) = \big[\alpha_i h(t) + \ell_i(t)\big]\frac{\delta(t)}{\delta_i(t)} \;\; (i=1,\dots,n-1),$$

α_i $(i=1,\dots,n-1)$ are nonnegative constants, while $h:]a,b]\to[0,+\infty[$, $\ell_i:]a,b]\to[0,+\infty[$ $(i=1,\dots,n-1)$ and $f_0:]a,b]\times\mathbb{R}^n\to[0,+\infty[$ are continuous functions such that

$$\int\limits_a^b h(t)\,dt=+\infty,\quad \int\limits_a^b \ell_i(t)\,dt<+\infty \ \ (i=1,\dots,n-1),$$

$$\alpha_0 h(t)\delta(t)\le f_0(t,x_1,\dots,x_n)\le \alpha h(t)\delta(t),\ \ \alpha>\alpha_0>0.$$

From Theorems 1 and 2 it follows

Corollary 1. *Problem* $(9),(2)$ *is solvable if and only if*

$$\sum_{i=1}^{n-1}\alpha_i<1.$$

Consequently, inequality (6) in Theorem 1 is unimprovable and it cannot be replaced by the nonstrict inequality

$$\limsup_{t\to a}\bigg[\frac{1}{\delta(t)}\sum_{i=1}^{n-1}\int\limits_a^t \exp\bigg(\int\limits_s^t h_n(\tau)\,d\tau\bigg)\delta_i(s)h_i(s)\,ds\bigg]\le 1.$$

Theorem 3. *If along with* (5) *conditions* (6)*–*(8) *are fulfilled, where* $h_0(t)=|f(t,0,\dots,0)|$*, then problem* $(1),(2)$ *has one and only one solution.*

Conditions (6)–(8) are satisfied, for example, if

$$\delta(t)=(t-a)^\lambda,\ \ \lambda\in\,]-1,+\infty[\,,\quad h_n(t)=\frac{\lambda}{t-a}-\exp\Big(\frac{1}{t-a}\Big),$$
$$h_i(t)=\alpha_i(t-a)^{i+1-n}\exp\Big(\frac{1}{t-a}\Big)\ \ (i=1,\dots,n-1),\quad f(t,0,\dots,0)=\alpha_0(t-a)^\lambda\exp\Big(\frac{1}{t-a}\Big).$$

Consequently, Theorem 3 covers the case, where the functions $t\mapsto h_i(t)$ $(i=1,\dots,n)$ and $t\mapsto f(t,0,\dots,0)$ have singularities of arbitrary order for $t=a$.

The following theorem contains the conditions guaranteeing the existence of an infinite set of solutions of problem $(1),(2)$. It concerns the case, when on the set $]a,b]\times\mathbb{R}^n$ the inequality

$$-\sum_{i=1}^{n-1}h_i(t)|x_i|-h_0(t)\le \big(f(t,x_1,\dots,x_n)-h_n(t)x_n\big)\operatorname{sgn}(x_n)\le\sum_{i=1}^{n-1}\overline{h}_i(t)|x_i|+\overline{h}_0(t)\tag{10}$$

is satisfied, where $h_i:]a,b]\to\mathbb{R}_+$, $\overline{h}_i:]a,b]\to\mathbb{R}_+$ $(i=0,\dots,n-1)$ and $h_n:]a,b]\to\mathbb{R}$ are continuous functions.

Theorem 4. *If along with* (10) *the conditions*

$$\int\limits_a^b \exp\bigg(\int\limits_s^b h_n(\tau)\,d\tau\bigg)\delta_i(s)h_i(s)\,ds<+\infty \ \ (i=1,\dots,n-1),$$

$$\int\limits_a^b \exp\bigg(\int\limits_s^b h_n(\tau)\,d\tau\bigg)h_0(s)\,ds<+\infty,\quad \liminf_{t\to a}\bigg[\delta(t)\exp\bigg(\int\limits_t^b h_n(\tau)\,d\tau\bigg)\bigg]>0$$

are fulfilled, then problem $(1),(2)$ *has an infinite set of solutions.*

Finally, let us consider the linear differential equation

$$u^{(n)}=\sum_{i=1}^{n}p_i(t)u^{(i-1)}+p_0(t) \tag{11}$$

with continuous coefficients $p_i:\,]a,b]\to\mathbb{R}$ $(i=0,1,\dots,n)$.

From Theorems 1, 3, 4 we have the following corollaries.

Corollary 2. *If*

$$\limsup_{t\to a}\bigg[\frac{1}{\delta(t)}\sum_{i=1}^{n-1}\int\limits_a^t\exp\bigg(\int\limits_s^t p_n(\tau)\,d\tau\bigg)\delta_i(s)|p_i(s)|\,ds\bigg]<1, \tag{12}$$

$$\limsup_{t\to a}\bigg[\frac{1}{\delta(t)}\int\limits_a^t\exp\bigg(\int\limits_s^t p_n(\tau)\,d\tau\bigg)|p_0(s)|\,ds\bigg]<+\infty, \tag{13}$$

then problem (11), (2) *has at least one solution. If, however, along with* (12) *and* (13) *the condition*

$$\liminf_{t\to a}\bigg[\delta(t)\exp\bigg(\int\limits_t^b h_n(s)\,ds\bigg)\bigg]=0$$

is fulfilled, then this problem has a unique solution.

Corollary 3. *Let the function* $t\mapsto\delta(t)\exp\big(\int_t^b p_n(s)\,ds\big)$ *be nondecreasing and*

$$\int\limits_a^b\frac{\delta_i(s)}{\delta(s)}\,|p_i(s)|\,ds<+\infty\ \ (i=1,\dots,n-1),\quad \int\limits_a^b\frac{|p_0(s)|}{\delta(s)}\,ds<+\infty.$$

Then problem (11), (2) *is uniquely solvable if and only if*

$$\lim_{t\to a}\bigg[\delta(t)\exp\bigg(\int\limits_t^b p_n(s)\,ds\bigg)\bigg]=0.$$

If, however,

$$\lim_{t\to a}\bigg[\delta(t)\exp\bigg(\int\limits_t^b p_n(s)\,ds\bigg)\bigg]>0,$$

then this problem has an infinite set of solutions.

The strict inequality (12) in Corollary 2 is unimprovable and it cannot be replaced by the nonstrict one.

A particular case of (2) is the condition

$$\lim_{t\to a}u^{(i-1)}(t)=0\ \ (i=1,\dots,n-1),\quad \limsup_{t\to a}\frac{|u^{(n-1)}(t)|}{(t-a)^{\lambda}}<+\infty, \tag{14}$$

where $\lambda\in\,]-1,+\infty[\,$.

Corollary 4. *Let*

$$p(t) \overset{def}{=} \frac{\lambda}{t-a} - p_n(t) > 0 \ \ for \ \ a < t \leq b, \tag{15}$$

$$\lim_{t\to a} \frac{(t-a)^{n-i} p_i(t)}{p(t)} = 0, \ \ (i=1,\dots,n-1), \quad \limsup_{t\to a} \frac{|p_0(t)|}{(t-a)^{\lambda} p(t)} < +\infty. \tag{16}$$

Then problem (11), (14) *is uniquely solvable if and only if*

$$\int_a^b p(t)\,dt = +\infty. \tag{17}$$

Remark. If $\lambda > 0$, then, obviously, the conditions of Corollary 4 guarantee the existence of a solution of equation (11) satisfying the initial conditions

$$\lim_{t\to a} u^{(i-1)}(t) = 0 \ \ (i=1,\dots,n). \tag{18}$$

On the other hand, if $\lambda \in]-1,0]$ and along with (16)–(18) the condition

$$\liminf_{t\to a} \frac{|p_0(t)|}{(t-a)^{\lambda} p(t)} > 0$$

is fulfilled, then problem (11), (18) has no solution, whereas problem (11), (14) is uniquely solvable.