Existence of Optimal Controls for Some Classes of Functional-Differential Equations

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Sufficient conditions for the existence of optimal controls for system of functional-differential equations which is nonlinear by phase variables and linear by control function are given. These conditions are obtained in terms of right-hand sides of the system and of the quality criterion function, which makes them convenient for verification.

Let $h > 0$ be a value of delay, $|\cdot|$ denotes the norm of the vector in the space $\mathbb{R}^d$, $\|\cdot\|$ be the norm of $d \times m$-dimensional matrix which is consistent with the norm of the vector. Let us denote as $C = C([-h,0];\mathbb{R}^d)$ the Banach space of continuous maps of $[-h,0]$ into $\mathbb{R}^d$ with the uniform norm $\|\varphi\|_C = \max\limits_{\theta\in[-h,0]} |\varphi(\theta)|$. Also denote as $L_p = L_p([-h,0];\mathbb{R}^m)$, $p > 1$, the Banach space of p-integrable m-dimensional vector-functions with standard norm

$$\|\varphi\|_{L_p} = \bigg(\int\limits_{-h}^{0} |\varphi(\tau)|^p \, d\tau \bigg)^{\frac{1}{p}}.$$

Let $x \in C([0,T];R^d)$, $\varphi \in C$. If $x(0) = \varphi(0)$, then the function

$$x(t,\varphi) = \begin{cases} \varphi(t), & t \in [-h,0], \\ x(t), & t \geq 0 \end{cases}$$

is continuous on $[-h,T]$. For each $t \in [0,T]$ in the standard way by $\theta \in [-h,0]$ we put an element $x_t(\varphi) \in C$ as $x_t(\varphi) = x(t+\theta,\varphi)$. In what follows we shall write x_t instead of $x_t(\varphi)$. Let $t \in [0,T]$, D is some domain in $[0,T] \times C$, ∂D is its boundary and $\overline{D} = D \cup \partial D$.

Now we consider the optimal control problems for systems of functional differential equations:

$$\dot{x} = f_1(t,x_t) + \int\limits_{-h}^{0} f_2(t,x_t,y)u(t,y)\,dy, \quad t \in [0,\tau], \tag{1}$$

$$x(t) = \varphi_0(t), \quad t \in [-h,0],$$

with the quality criterion

$$J[u] = \int\limits_{0}^{\tau} L(t,x_t,u(t,\cdot\,))\,dt \longrightarrow \inf \tag{2}$$

on $[0,T]$, where $\varphi_0 \in C$ is a fixed element such that $(0,\varphi_0) \in D$, $x(t)$ is the phase vector in $\mathbb{R}^d$, x_t is the phase vector in C, τ is the moment of the first exit (t,x_t) on the boundary ∂D, $f_1 : D \to \mathbb{R}^d$, $f_2 : D \times [-h,0] \to M^{d\times m}$ are $d \times m$-dimensional matrices, and for each $(t,\varphi) \in D$, $f_2(t,\varphi,\cdot\,) \in L_q([-h,0];M^{d\times m})$ with the norm

$$\|f_2(t,\varphi,\cdot\,)\|_{L_q} = \bigg(\int\limits_{-h}^{0} \|f_2(t,\varphi,y)\|^q \, dy \bigg)^{\frac{1}{q}},$$

$\frac{1}{q}+\frac{1}{p}=1$, $L : D\times L_p \to \mathbb{R}^1$. The control parameter $u\in L_p([0,T]\times[-h,0])$ is such that $u(t,y)\in U$, and U is convex and closed set in $\mathbb{R}^m$ for almost all t ,y.

Many works are devoted to the optimal control problems for functional-differential equations systems: the monograph [8] is devoted to the application of the method of dynamic programming and the principle of maximum to problems with aftereffect. In the case of compactness of the set of admissible controls in work [1] it was obtained an analogue of Filippov theorem of optimality control existence for ordinary differential equations. For noncompact set of admissible controls an analogue of the Cessari theorem is obtained in [5]. In the mentioned work the condition of compactness is imposed on a set of constraints and a certain condition of growth which connects the right sides of the system and the quality criterion. In [2] under the condition of compactness of the set of admissible controls value sufficient conditions for optimality on a fixed interval $[t_0,t_1]$ for neutral-type equations are obtained. In [3] the problem of optimal control of a delayed linear system is rewritten in a form that does not depend on the delay, which is studied by methods of ordinary differential equations. In the works [4, 6, 7] the optimal control problem of the system

$$\dot{x}(t)=rx(t)+f_0\bigg(x(t),\int\limits_{-T}^{0}a(\varsigma)x(t+\varsigma)\,d\varsigma\bigg)-u(t)$$

is considered. In [4] certain Hamilton–Jacobi–Bellman equations are obtained for certain quality functionals and, in terms of their solutions, sufficient conditions for optimality in the form of a reverse link are obtained. In [6] similar questions are considered for problems with phase restriction. In [7] for such problems it was obtained sufficient conditions for optimality under the condition of nondecreasing function $rx+f_0(x,y)$ in both the variables.

The main goal of our paper is to obtain the theorem on the existence of optimal controls for a wider class of problems under weaker conditions as compared with above mentioned works [1–8]. The following are the main conditions for the problem (1), (2) assumed in the manuscript.

Assumption 1. *Admissible controls are m-dimensional vector functions $u\in L_p([0,T][-h,0],\mathbb{R}^m)$, such that $u(t,y)\in U$ for almost all $t\in[0,T]$ and $y\in[-h,0]$.*

The set of admissible controls we denote as $\mathcal{U}$.

Assumption 2. *Maps $f_1(t,\varphi): D\to\mathbb{R}^d$ and $f_2(t,\varphi,y): D\times[-h,0]\to M^{d\times m}$ are defined and measurable with respect to all its arguments in the domains D and $D_1=\{(t,\varphi)\in D,\ y\in[-h,0]\}$ respectively, satisfy the linear growth condition and the Lipschitz condition with respect to φ, i.e. there exists the constant $K>0$ such that*

$$|f_1(t,\varphi)|+\|f_2(t,\varphi,y)\|\le K(1+\|\varphi\|_C) \tag{3}$$

for any $(t,\varphi)\in D$, $y\in[-h,0]$,

$$\big|f_1(t,\varphi_1)-f_1(t,\varphi_2)\big|+\big\|f_2(t,\varphi_1,y)-f_2(t,\varphi_2,y)\big\|\le K\|\varphi_1-\varphi_2\|_C \tag{4}$$

for all $(t,\varphi_1),(t,\varphi_2)\in D$, $y\in[-h,0]$.

Assumption 3. *Conditions for the criterion function:*

1) *the map $L(t,\varphi,z): D\times L_p\to\mathbb{R}^1$ is defined and continuous with respect to all its arguments in the domain $D_2=\{(t,\varphi)\in D,\ z\in L_p\}$;*

2) *there exists $a>0$ such that*

$$\big|L(t,\varphi_1,z)-L(t,\varphi_2,z)\big|\le a\|\varphi_1-\varphi_2\|_C$$

for all $(t,\varphi_1,z),(t,\varphi_2,z)\in D_2$;

3) *Frechet derivative L_u of the map L is continuous with respect to all its arguments in the domain D_2, and there exist constants $C_1 > 0$, $\alpha > 0$ such that for all $(t, \varphi, z) \in D_2$ the following inequality holds:*

$$\|L_u(t, \varphi, z)\|_{L_q} \le C_1\big(1 + \|\varphi\|_C^\alpha + \|z\|_{L_p}^{p-1}\big);$$

4) *there exists the constant $C > 0$ such that $L(t, \varphi, z) \ge C\|z\|_{L_p}^p$ for all $(t, \varphi, z) \in D_2$;*

5) *$L(t, \varphi, z)$ convex with respect to z for any fixed t, φ;*

Our first result concerns the existence, uniqueness and extension of the solution of the original problem (1) to the boundary ∂D of the domain D. It is some analogue of the Carathéodory theorem for ordinary differential equations.

Definition. The solution of the initial problem (1) on the segment $[-h, A]$, $A > 0$ is called a continuous on the segment $[-h, A]$ function $x(t)$ such that

1) $x(t) = \varphi_0(t)$, $t \in [-h, 0]$;
2) $(t, x_t) \in D$ on $t \in [0, A]$;
3) for $t \in [0, A]$ function $x(t)$ satisfies the integral equation

$$x(t) = \varphi_0(0) + \int_0^t \Big[f_1(s, x_s) + \int_{-h}^0 f_2(s, x_s, y)u(s, y)\,dy\Big]\,ds. \tag{5}$$

Remark. It is obvious that for $t \in [0, A]$ solution $x(t)$ is an absolutely continuous function and satisfies the equation (1) for almost all t on $[0, A]$.

Theorem 1. *Suppose that Assumptions* 1, 2 *are satisfied. Then there exists the solution of the initial problem* (5) *on the maximal segment $[-h, \tau]$, $\tau > 0$ and $(\tau, x_\tau) \in \partial D$.*

The following theorem gives for the problem (1), (2) the existence conditions of the optimal pair $x^*(t)$, $u^*(t, \theta)$), which provides the minimum of the quality criterion (2). In this case $u^* \in \mathcal{U}$ is called optimal control and the corresponding trajectory $x^*(t)$ (1) is called the optimal trajectory.

Theorem 2. *Suppose that Assumptions* 1–3 *are satisfied. Then there exists the solution of the optimal control problem* (1), (2).

As an application of the obtained results, we consider some particular cases of problem (1), (2). If $u = u(t)$ and does not depend on the value y, then the problem (1), (2) reduces to the "ordinary" optimal control problem for functional-differential equations. A particular case of the problem (1), (2) is the optimal control problem with maximum on the interval $[-h, T]$, $h > 0$.

$$\dot{x}(t) = f_1\big(t, x_t, \max_{s\in I(t)} x(s)\big) + f_2\big(t, x_t, \max_{s\in I(t)} x(s)\big)u(t), \tag{6}$$

$$x(t) = \varphi(t), \quad t \in [-h, 0],$$

$$J[u] = \int_0^\tau L(t, x(t), u(t))\,dt \longrightarrow \inf, \tag{7}$$

where $I(t) = [\beta(t), \alpha(t)]$, $\max x(s) = (\max x_1(s), \dots, \max x_d(s))$, $\beta(t)$, $\alpha(t)$ are continuous on $[0, T]$ functions such that $\beta(t) \le \alpha(t) \le t$ and $\min\limits_{t\in[0,T]} (\beta(t) - t) = -h$, $f(t, x, y) : [0, T] \times G \times G \to M^{d\times m}$, G is a domain in $\mathbb{R}^d$, $u \in U \subset \mathbb{R}^m$, $L(t, x, u) : [0, T] \times G \times U \to \mathbb{R}^1$.

References

[1] T. S. Angell, Existence theorems for optimal control problems involving functional differential equations. *J. Optimization Theory Appl.* **7** (1971), 149–169.

[2] H. T. Banks and G. A. Kent, Control of functional differential equations of retarded and neutral type to target sets in function space. *SIAM J. Control* **10** (1972), 567–593.

[3] G.-P. Cai, J.-Z. Huang and S. X. Yang, An optimal control method for linear systems with time delay. *Comput. & Structures* **81** (2003), no. 15, 1539–1546.

[4] G. Carlier and R. Tahraoui, On some optimal control problems governed by a state equation with memory. *ESAIM Control Optim. Calc. Var.* **14** (2008), no. 4, 725–743.

[5] L. Cesari, Existence theorems for weak and usual optimal solutions in Lagrange problems with unilateral constraints. I. *Trans. Amer. Math. Soc.* **124** (1966), 369–412; II. Existence theorems for weak solutions. *Trans. Amer. Math. Soc.* **124** (1966), 413–430.

[6] S. Federico, B. Goldys and F. Gozzi, HJB equations for the optimal control of differential equations with delays and state constraints, I: regularity of viscosity solutions. *SIAM J. Control Optim.* **48** (2010), no. 8, 4910–4937.

[7] S. Federico, B. Goldys and F. Gozzi, HJB equations for the optimal control of differential equations with delays and state constraints, II: Verification and optimal feedbacks. *SIAM J. Control Optim.* **49** (2011), no. 6, 2378–2414.

[8] V. B. Kolmanovskiĭ and L. E. Shaĭkhet, *Control of Systems with Aftereffect.* (Russian) Nauka, Moscow, 1992; translations in Mathematical Monographs, 157. American Mathematical Society, Providence, RI, 1996.