Positive Invertible Matrices and Stability of Nonlinear Itô Equations

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Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ be a stochastic basis consisting of a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and an increasing, right-continuous family (a filtration) $(\mathcal{F}_t)_{t\geq 0}$ of complete σ-subalgebras of $\mathcal{F}$. By E we denote the expectation on this probability space. The scalar stochastic processes $\mathcal{B}_i$, $i=2,\dots,m$ are scalar, independent Brownian motions on $(\mathcal{F}_t)_{t\geq 0}$ (see e.g. [6]).

The following inequality holds true for any Brownian motion $\mathcal{B}(s)$ and any scalar stochastic process $f(s)$, which is integrable with respect to $\mathcal{B}(s)$ on $[0,t]$:

$$\bigg(E\bigg|\int\limits_0^t f(s)\,d\mathcal{B}(s)\bigg|^{2p}\bigg)^{1/2p}\leq c_p\bigg(E\bigg(\int\limits_0^t |f(s)|^2\,ds\bigg)^p\bigg)^{1/2p}.$$

Here c_p is some number depending on p. Some estimates on this number can be found e.g. in [6].

We consider the following system of Itô equations with delay:

$$\begin{aligned} dx_i(t)=\Big[-a_i(t)x_i(h_i(t))+\sum_{j=1}^n F_{ij}(t,x_j(h_{ij}(t)))\Big]\,dt\\ +\sum_{l=1}^m\Big[\sum_{j=1}^n G_{ij}^l\big(t,x_j(h_{ij}^l(t))\big)\Big]\,d\mathcal{B}_l(t)\ \ (t\geq 0),\ \ i=1,\dots,n \end{aligned}\tag{1}$$

with the initial conditions

$$x_i(t)=\varphi_i(t)\ \ (t<0),\ \ i=1,\dots,n,, \tag{1_a}$$

$$x_i(t)=b_i,\ \ i=1,\dots,n, \tag{1_b}$$

where

1) a_i are Lebesgue measurable functions, which are defined on $[0,\infty)$ and satisfy $0<\overline{a}_i\leq a_i\leq A_i$ $(t\in[0,\infty))$ μ-everywhere for some positive numbers $\overline{a}_i$, A_i $(i=1,\dots,n)$;

2) $F_{ij}(\,\cdot\,,u)$ are Lebesgue measurable functions defined on $[0,\infty)$, $F_{ij}(t,\,\cdot\,)$ are continuous functions, which are defined on R^1 and satisfy $|F_{ij}(t,u)|\leq\overline{F}_{ij}|u|$ $(t\in[0,\infty))$ μ-everywhere for some positive numbers $\overline{F}_{ij}$ $(i,j=1,\dots,n)$;

3) $G_{ij}^l(\,\cdot\,,u)$ are Lebesgue measurable functions defined on $[0,\infty)$, $G_{ij}^l(t,\,\cdot\,)$ are continuous functions, which are defined on R^1 and satisfy $|G_{ij}^l(t,u)|\leq\overline{G}_{ij}^l|u|$ $(t\in[0,\infty))$ μ-everywhere for some positive numbers $\overline{G}_{ij}^l$ $(l=1,\dots,m;\ i,j=1,\dots,n)$;

4) h_i, h_{ij}, h_{ij}^l are Borel measurable functions defined on $[0,\infty)$ and satisfy $0 \le t - h_i(t) \le \tau_i$, $0 \le t - h_{ij}(t) \le \tau_{ij}$, $0 \le t - h_{ij}^l(t) \le \tau_{ij}^l$ $(t \in [0,\infty))$ μ-everywhere for some positive numbers τ_i, τ_{ij}, τ_{ij}^l for $l = 1,\dots,m$; $i,j = 1,\dots,n$;

5) φ_i are $\mathcal{F}_0$-measurable scalar stochastic processes defined on $[\sigma_i, 0)$, where $\sigma_i = \max\{\tau_i, \tau_{ij}, \tau_{ij}^l,$ $l = 1,\dots,m;\ j = 1,\dots,n\}$;

6) b_i are $\mathcal{F}_0$-measurable scalar random values $(i = 1,\dots,n)$.

We remark that the initial value problem $(1),(1_a),(1_b)$ has a unique solution if the functions $F_{ij}(t,u)$, $G_{ij}^l t,u)$ are Lipschits with respect to u for all $l = 1,\dots,m$, $i,j = 1,\dots,n$ (see e. g. [3]). In what follows, we assume that this is the case and denote by $x(t,b,\varphi)$ the solution of (1) satisfying (1_a) and (1_b), so that $x(s,b,\varphi) = \varphi$ for $s < 0$ and $x(0,b,\varphi) = b$.

Definition 1. For a given real number p $(1 \le p < \infty)$ we say that system (1) is globally exponentially p-stable (w.r.t. the initial data) if there exist positive constants $\overline{c}$, β such that the inequality

$$E|x(t,x_0,\varphi)|^p \le \overline{c}\left(E|x_0|^p + \operatorname*{ess\,sup}_{s<0} E|\varphi(s)|^p\right)\exp\{-\beta s\}$$

holds true for all $t \ge 0$ and all φ, x_0.

An $n \times n$-matrix $\Gamma = (\gamma_{ij})_{i,j=1}^n$ is called nonnegative if $\gamma_{ij} \ge 0$, $i,j = 1,\dots,n$, and positive if $\gamma_{ij} > 0$, $i,j = 1,\dots,n$.

Definition 2. A matrix $\Gamma = (\gamma_{ij})_{i,j=1}^n$ is called an $\mathcal{M}$-matrix if $\gamma_{ij} \le 0$ for $i,j = 1,\dots,n$, $i \ne j$ and one of the following conditions is satisfied:

- Γ has a positive inverse matrix Γ^{-1};
- the principal minors of the matrix Γ are positive.

Below we define the $n \times n$-matrix Γ in the following way

$$\gamma_{ii} = 1 - \frac{A_i^2\tau_i^2 + A_i\overline{F}_{ii}\tau_i + c_pA_i\sqrt{\tau_i}\sum\limits_{i=1}^m \overline{G}_{ii}^l + \overline{F}_{ii}}{\overline{a}_i} - \frac{c_p\sum\limits_{l=1}^m \overline{G}_{ii}^l}{\sqrt{2\overline{a}_i}}\,, \quad i = 1,\dots,n,$$

$$\gamma_{ij} = -\frac{A_i\overline{F}_{ij}\tau_i + c_pA_i\sqrt{\tau_i}\sum\limits_{i=1}^m \overline{G}_{ij}^l + \overline{F}_{ij}}{\overline{a}_i} - \frac{c_p\sum\limits_{l=1}^m \overline{G}_{ij}^l}{\sqrt{2\overline{a}_i}}\,, \quad i,j = 1,\dots,n, \quad i \ne j.$$

Theorem. *If the matrix Γ defined above is an $\mathcal{M}$-matrix, then system* (1) *is globally exponentially $2p$-stable.*

Outline of the proof (see [5] for the details).

The main idea is to use the W-method (see [1, 3, 4] and the references therein) to regularize system (1) to obtain a certain integral operator in a suitable space of stochastic processes. This operator can be constructed with the help of an auxiliary linear equation, which is similar to the equation (1):

$$dx(t) = \big[(Qx)(t) + g(t)\big]\,dZ(t), \quad t \ge 0,$$

The solutions of this equation has the Cauchy representation

$$x(t) = U(t)x_0 + (Wg)(t), \quad t \ge 0,$$

where $U(t)$ is the fundamental matrix of the associated homogeneous equation, and W is the corresponding Cauchy operator.

Assuming for the sake of simplicity that system (1) is also linear, rewriting it in the operator form

$$dx(t) = \big[(Vx)(t) + f(t)\big]\,dZ(t), \ \ t \geq 0$$

and substituting the above Cauchy representation formula into this equation result in

$$dx(t) = \big[(Qx)(t) + ((V-Q)x)(t) + f(t)\big]\,dZ(t), \ \ t \geq 0,$$

or

$$x(t) = U(t)x(0) + (W(V-Q)x)(t) + (Wf)(t), \ \ t \geq 0.$$

Denoting $W(V-Q) = \Theta$, we obtain the operator equation $((I-\Theta)x)(t) = U(t)x(0) + (Wf)(t)$. If now the operator $I-\Theta$ is invertible in a suitable space of stochastic processes, then system (1) is globally asymptotically $2p$-stable.

In most implementations of this scheme, one tries to prove that the norm of the operator Θ is less than 1. Then $I-\Theta$ becomes invertible.

However, this approach may lead to too rough estimates. A more careful approach, based on the theory of positive matrices, was suggested in [2], where straight invertibility in norm is replaced by matrix inequalities. In particular, if the corresponding matrix is an $\mathcal{M}$-matrix, then we still can prove the global asymptotic $2p$-stability of system (1).

This approach is utilized in the paper [5] as well as in this presentation in the case of stochastic functional differential equations.

Let us now study system (1) in two dimensions.

Corollary 1. *Let $n=2$ in system* (1) *and*

$$\sqrt{2}\,\Big(A_1^2\tau_1^2 + A_1\overline{F}_{11}\tau_1 + c_pA_1\sqrt{\tau_1}\sum_{i=1}^{m}\overline{G}_{11}^l + \overline{F}_{11}\Big) + \sqrt{\overline{a}_1}\,c_p\sum_{l=1}^{m}\overline{G}_{11}^l < \sqrt{2}\,\overline{a}_1,$$

$$\begin{aligned}
&\bigg(\sqrt{2}\overline{a}_1 - \sqrt{2}\,\Big(A_1^2\tau_1^2 + A_1\overline{F}_{11}\tau_1 + c_pA_1\sqrt{\tau_1}\sum_{i=1}^{m}\overline{G}_{11}^l + \overline{F}_{11}\Big) - \sqrt{\overline{a}_1}c_p\sum_{l=1}^{m}\overline{G}_{11}^l\bigg)\\
&\quad\times\bigg(\sqrt{2}\overline{a}_2 - \sqrt{2}\,\Big(A_2^2\tau_2^2 + A_2\overline{F}_{22}\tau_2 + c_pA_2\sqrt{\tau_2}\sum_{i=1}^{m}\overline{G}_{22}^l + \overline{F}_{22}\Big) - \sqrt{\overline{a}_2}c_p\sum_{l=1}^{m}\overline{G}_{22}^l\bigg)\\
&> \bigg(\sqrt{2}\,\Big(A_1\overline{F}_{12}\tau_1 + c_pA_1\sqrt{\tau_1}\sum_{i=1}^{m}\overline{G}_{12}^l + \overline{F}_{12}\Big) + \sqrt{\overline{a}_1}c_p\sum_{l=1}^{m}\overline{G}_{12}^l\bigg)\\
&\quad\times\bigg(\sqrt{2}\Big(A_2\overline{F}_{21}\tau_2 + c_pA_2\sqrt{\tau_2}\sum_{i=1}^{m}\overline{G}_{21}^l + \overline{F}_{21}\Big) + \sqrt{\overline{a}_2}c_p\sum_{l=1}^{m}\overline{G}_{21}^l\bigg).
\end{aligned}$$

Then system (1) *is globally exponentially* $2p$*-stable.*

Proof. We exploit the main theorem. Under the assumptions of Corollary 1 the matrix Γ becomes 2×2 with nonnegative off-diagonal entries. Thus, it will become an $\mathcal{M}$-matrix, if its principal minors γ_{11} and $\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21}$ are positive. Straightforward calculations show that the first inequality of Corollary 1 yields $\gamma_{11} > 0$, while the second inequality of Corollary 1 yields $\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21} > 0$.

The corollaries below can be proven in a similar way. □

Corollary 2. *Consider the system*

$$\begin{aligned} dx_i(t) = \Big[-a_i(t)x_i(t) + \sum_{j=1}^{n} F_{ij}(t)x_j(h_{ij}(t))\Big]\,dt \\ + \sum_{l=1}^{m} \Big[\sum_{j=1}^{n} G^l_{ij}(t)x_j(h^l_{ij}(t))\Big]\,d\mathcal{B}_l(t) \ \ (t \ge 0), \ \ i = 1,\dots,n. \end{aligned} \tag{2}$$

with $n = 2$ assuming that

$$\sqrt{2}\,\overline{F}_{11} + \sqrt{\overline{a}_1}\,c_p \sum_{l=1}^{m} \overline{G}^l_{11} < \sqrt{2}\,\overline{a}_1,$$

$$\begin{aligned} \Big(\sqrt{2}\,\overline{a}_1 - \sqrt{2}\,\overline{F}_{11} - \sqrt{\overline{a}_1}\,c_p \sum_{l=1}^{m} \overline{G}^l_{11}\Big)\Big(\sqrt{2}\,\overline{a}_2 - \sqrt{2}\,\overline{F}_{22} - \sqrt{\overline{a}_2}\,c_p \sum_{l=1}^{m} \overline{G}^l_{22}) \\ > \Big(\sqrt{2}\,\overline{F}_{12} + \sqrt{\overline{a}_1}\,c_p \sum_{l=1}^{m} \overline{G}^l_{12}\Big)\Big(\sqrt{2}\,F_{21} + \sqrt{\overline{a}_2}\,c_p \sum_{l=1}^{m} \overline{G}^l_{21}\Big). \end{aligned}$$

Then system (2) *is globally asymptotically* $2p$*-stable.*

Corollary 3. *Consider the system*

$$\begin{aligned} dx_i(t) = \Big[-a_i(t)x_i(t) + \sum_{j=1,\,i\neq j}^{n} F_{ij}(t)x_j(h_{ij}(t))\Big]\,dt \\ + \sum_{l=1}^{m} \Big[\sum_{j=1,\,i\neq j}^{n} G^l_{ij}(t)x_j(h^l_{ij}(t))\Big]\,d\mathcal{B}_l(t) \ \ (t \ge 0), \ \ i = 1,\dots,n. \end{aligned} \tag{3}$$

with $n = 2$ assuming that

$$\Big(\sqrt{2}\,\overline{F}_{12} + \sqrt{\overline{a}_1}\,c_p \sum_{l=1}^{m} \overline{G}^l_{12}\Big)\Big(\sqrt{2}\,F_{21} + \sqrt{\overline{a}_2}\,c_p \sum_{l=1}^{m} \overline{G}^l_{21}\Big) < 2\overline{a}_1\overline{a}_2.$$

Then system (3) *is globally asymptotically* $2p$*-stable.*

Example 1. Consider the system

$$\begin{aligned} dx_1(t) = \Big[-a_1x_1(t-h_1) + a_{11}F_{11}(x_1(t-h_{11})) + a_{12}F_{12}(x_2(t-h_{12}))\Big]\,dt \\ + \Big[b_{11}G_{11}(x_1(t-\tau_{11})) + b_{12}G_{12}(x_2(t-\tau_{12}))\Big]\,d\mathcal{B}(t) \ \ (t \ge 0), \\ dx_2(t) = \Big[-a_2x_1(t-h_2) + a_{21}F_{21}(x_1(t-h_{21})) + a_{22}F_{22}(x_2(t-h_{22}))\Big]\,dt \\ + \Big[b_{21}G_{21}(x_1(t-\tau_{21})) + b_{22}G_{22}(x_2(t-\tau_{22}))\Big]\,d\mathcal{B}(t) \ \ (t \ge 0), \end{aligned} \tag{4}$$

where a_1, a_2, h_{ij}, τ_{ij}, a_{ij}, b_{ij}, $i,j = 1,2$ are positive numbers, F_{ij}, G_{ij}, $i,j = 1,2$ are continuous scalar functions on $(-\infty,+\infty)$ such that $|F_{ij}(u)| \le |u|$, $|G_{ij}(u)| \le |u|$, $i,j = 1,2$, and $\mathcal{B}$ is the standard scalar Brownian motion.

Then from Corollary 1 we deduce that the conditions

$$\sqrt{2}\,\Big(a_1^2h_1^2 + a_1a_{11}h_1 + c_pa_1\sqrt{h_1}\,b_{11} + a_{11}\Big) + \sqrt{a_1}\,c_pb_{11} < \sqrt{2}\,a_1,$$

$$\begin{aligned}\Big(\sqrt{2}\,a_1-\sqrt{2}\,\Big(a_1^2h_1^2+a_1a_{11}h_1+c_pa_1\sqrt{h_1}\,b_{11}+a_{11}\Big)-\sqrt{a_1}\,c_pb_{11}\Big)&\\ \times\Big(\sqrt{2}\,a_2-\sqrt{2}\,\Big(a_2^2h_2^2+a_2a_{22}h_2+c_pa_2\sqrt{h_2}\,b_{22}+a_{22}\Big)-\sqrt{a_2}\,c_pb_{22}\Big)&\\ >\Big(\sqrt{2}\,(a_1a_{12}h_1+c_pa_1\sqrt{h_1}\,b_{12}+a_{12})+\sqrt{a_1}\,c_pb_{12}\Big)&\\ \times\Big(\sqrt{2}\,(a_2a_{21}h_2+c_pa_2\sqrt{h_2}\,b_{21}+a_{21})+\sqrt{a_2}\,c_pb_{21}\Big)&\end{aligned}$$

imply the global asymptotic $2p$-stability of system (4).

Assume further that $a_{ii}=b_{ii}=0$, $i=1,2$ in system (4). In this case, the conditions

$$a_1h_1^2<1,$$

$$\begin{aligned}\Big(\sqrt{2}\,a_1-\sqrt{2}\,a_1^2h_1^2\Big)\Big(\sqrt{2}\,a_2-\sqrt{2}\,a_2^2h_2^2\Big)>\Big(\sqrt{2}\,(a_1a_{12}h_1+c_pa_1\sqrt{h_1}\,b_{12}+a_{12})+\sqrt{a_1}\,c_pb_{12}\Big)&\\ \times\Big(\sqrt{2}\,(a_2a_{21}h_2+c_pa_2\sqrt{h_2}\,b_{21}+a_{21})+\sqrt{a_2}\,c_pb_{21}\Big)&\end{aligned}$$

imply the global asymptotic $2p$-stability of system (4).

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