# Structure of Nonoscillatory Solutions of First Order Nonlinear Differential Systems of Emden–Fowler Type

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## Section 1

We consider first order nonlinear cyclic differential systems of the form

$$x' - p(t)\varphi_\alpha(y) = 0, \quad y' + q(t)\varphi_\beta(x) = 0, \tag{A}$$

where $\alpha$ and $\beta$ are positive constants, $p$ and $q$ are positive continuous functions on $[0,\infty)$ and $\varphi_\gamma$, $\gamma > 0$, denotes the function

$$\varphi_\gamma(u) = |u|^\gamma \operatorname{sgn} u, \ \ u \in \mathbf{R}.$$

We are concerned exclusively with nonoscillatory solutions of (A), by which we mean those solutions $(x,y)$ whose components $x$ and $y$ are nonoscillatory in the usual sense.

Oscillation theory of systems of the form (A) was created by Mirzov, whose achievements of great theoretical interest are summarized in the monograph [2]. There are some topics untouched in [2], one of which is the systematic study of nonoscillatory solutions of (A). The present work, motivated by this observation, aims to depict a clear picture of the overall structure of nonoscillatory solutions of (A) by analyzing their asymptotic behavior at infinity as precisely as possible.

Let $(x,y)$ be a nonoscillatory solution of (A) on $[t_0,\infty)$. Since both $x$ and $y$ are eventually one-signed, they are monotone for all large $t$ so that the limits $x(\infty) = \lim\limits_{t\to\infty} x(t)$ and $y(\infty) = \lim\limits_{t\to\infty} y(t)$ exist in the extended real numbers. Thus, $x(t)y(t) \neq 0$ on $[T,\infty)$ for some $T \geq t_0$. We say that $(x,y)$ is a solution *of the first kind* (resp. *of the second kind*) if $x(t)y(t) > 0$ (resp. $x(t)y(t) < 0$) for $t \geq T$.

We use the notation

$$I_p = \int\limits_0^\infty p(t)\,dt, \quad I_q = \int\limits_0^\infty q(t)\,dt,$$

and examine the existence of nonoscillatory solutions of (A) by distinguishing the following four cases:

$$I_p = \infty \wedge I_q = \infty, \quad I_p = \infty \wedge I_q < \infty, \quad I_p < \infty \wedge I_q = \infty, \quad I_p < \infty \wedge I_q < \infty.$$

(The case $I_p = \infty \wedge I_q = \infty$) In this case, as is shown by Mirzov [2], all solutions of (A) are oscillatory, so that (A) has no nonoscillatory solutions.

(The case $I_p = \infty \wedge I_q < \infty$) In this case it can be shown that all nonoscillatory solutions $(x, y)$ are of the first kind, and that $|x|$ are eventually increasing and $|y|$ are eventually decreasing. Thus their asymptotic behavior as $t \to \infty$ can be classified into the three types

I(i): $|x(\infty)| = \infty$, $0 < |y(\infty)| < \infty$,

I(ii): $|x(\infty)| = \infty$, $|y(\infty)| = 0$,

I(iii): $0 < |x(\infty)| < \infty$, $|y(\infty)| = 0$.

Nonoscillatory solutions of the types I(i) and I(iii) are termed, respectively, *maximal solutions* and *minimal solutions* of the first kind of (A), and their existence can be characterized as the following theorem shows.

**Theorem 1.1.** *Let $\alpha$ and $\beta$ be any given positive constants.*

(i) *System* (A) *has solutions of the type* I(i) *if and only if*

$$\int_0^\infty q(t)P(t)^\beta \, dt < \infty, \quad \text{where } P(t) = \int_0^t p(s)\, ds.$$

(ii) *System* (A) *has solutions of the type* I(iii) *if and only if*

$$\int_0^\infty p(t)\rho(t)^\alpha \, dt < \infty, \quad \text{where } \rho(t) = \int_t^\infty q(s)\, ds.$$

Solutions of the type I(ii), which may be termed *intermediate solutions* of the first kind of (A), are so difficult to analyze that we have so far been able to handle only the system (A) whose nonlinearity is referred to as *sub-half-linear*.

**Theorem 1.2.** *Let $\alpha\beta < 1$. System* (A) *possesses a solution of the type* I(ii) *if and only if*

$$\int_0^\infty p(t)\rho(t)^\alpha \, dt < \infty \quad \text{and} \quad \int_0^\infty q(t)P(t)^\beta \, dt = \infty.$$

(The case $I_p < \infty \wedge I_q = \infty$) In this case it is shown that all nonoscillatory solutions $(x, y)$ are of the second kind, and that $|x|$ are eventually decreasing and $|y|$ are eventually increasing. Thus their asymptotic behavior as $t \to \infty$ can be classified into the three types

II(i): $0 < |x(\infty)| < \infty$, $|y(\infty)| = \infty$,

II(ii): $|x(\infty)| = 0$, $|y(\infty)| = \infty$,

II(iii): $|x(\infty)| = 0$, $0 < |y(\infty)| < \infty$.

Solutions of the types II(i) and II(iii) are called, respectively, *maximal solutions* and *minimal solutions* of the second kind of (A), while those of the type II(ii) are called *intermediate solutions* of the second kind of (A). As for the solutions of this kind we have the following results which are in parallel with Theorems 1.1 and 1.2 regarding the solutions of the first kind.

**Theorem 1.3.** *Let $\alpha$ and $\beta$ be any given positive constants.*

(i) *System* (A) *has solutions of the type* II(i) *if and only if*

$$\int_0^\infty p(t)Q(t)^\alpha\,dt < \infty, \quad \text{where } Q(t) = \int_0^t q(s)\,ds.$$

(ii) *System* (A) *has solutions of the type* II(iii) *if and only if*

$$\int_0^\infty q(t)\pi(t)^\beta\,dt < \infty, \quad \text{where } \pi(t) = \int_t^\infty p(s)\,ds.$$

**Theorem 1.4.** *Let $\alpha\beta < 1$. System* (A) *possesses a solution of the type* II(ii) *if and only if*

$$\int_0^\infty q(t)\pi(t)^\beta\,dt < \infty \ \text{ and } \int_0^\infty p(t)Q(t)^\alpha\,dt = \infty.$$

(The case $I_p < \infty \wedge I_q < \infty$) In this case it is shown without difficulty that all nonoscillatory solutions $(x,y)$ of (A) are bounded and that both $x$ and $y$ have finite limits as $t \to \infty$. As a matter of fact, for any given constants $c$ and $d$, zero or nonzero, there exists a nonoscillatory solution $(x,y)$ of (A) such that $x(\infty) = c$ and $y(\infty) = d$. Thus, (A) possesses solutions of both the first and second kinds.

## Section 2

We now turn our attention to scalar second order nonlinear differential equations of the form

$$(p(t)\varphi_\alpha(x'))' + q(t)\varphi_\beta(x) = 0, \tag{E}$$

where $\alpha$ and $\beta$ are positive constants, and $p$ and $q$ are positive continuous functions on $[0,\infty)$. Equation (E) is often called a *generalized Emden–Fowler equation* and has been the object of intensive investigation in its own right (see e.g., [1,3]).

We are interested in the structure of the totality of nonoscillatory solutions of equation (E). Let $x$ be a nonoscillatory solution of (E). We put $Dx = p(t)\varphi_\alpha(x')$ and call it the *quasi-derivative* of $x$. Worthy of note is the fact that by the substitution $y = Dx$ equation (E) is split into the differential system

$$x' - p(t)^{-\frac{1}{\alpha}}\varphi_{\frac{1}{\alpha}}(y) = 0, \quad y' + q(t)\varphi_\beta(x) = 0, \tag{B}$$

which can be regarded as a special case of system (A).

We say that a nonoscillatory solution $x$ of (E) is of the first kind or of the second kind if $x(t)Dx(t) > 0$ or $x(t)Dx(t) < 0$ for all large $t$, respectively.

If $p^{-\frac{1}{\alpha}}$ and $q$ satisfy

$$\int_0^\infty p(t)^{-\frac{1}{\alpha}}\,dt = \infty \ \text{ and } \int_0^\infty q(t)\,dt < \infty, \tag{2.1}$$

then all nonoscillatory solutions $x$ of (E) are of the first kind, and there are three possibilities for their asymptotic behavior at infinity

I(i): $|x(\infty)| = \infty$, $0 < |Dx(\infty)| < \infty$, I(ii): $|x(\infty)| = \infty$, $|Dx(\infty)| = 0$,

I(iii): $0 < |x(\infty)| < \infty$, $|Dx(\infty)| = 0$.

If $p^{-\frac{1}{\alpha}}$ and $q$ satisfy

$$\int\limits_0^\infty p(t)^{-\frac{1}{\alpha}}\,dt < \infty \ \text{ and } \int\limits_0^\infty q(t)dt = \infty, \tag{2.2}$$

then all nonoscillatory solutions $x$ of (E) are of the second kind, and there are three patterns of their asymptotic behavior at infinity

II(i): $0 < |x(\infty)| < \infty$, $|Dx(\infty)| = \infty$,

II(ii): $|x(\infty)| = 0$, $|Dx(\infty)| = \infty$,

II(iii): $|x(\infty)| = 0$, $0 < |Dx(\infty)| < \infty$.

In order to characterize the existence of solutions of these six types of (E) it suffices to specialize the results of Section 1 to system (B). For example, Theorems 1.2 and 1.4 applied to (B) which has to be sub-half-linear give rise to the following results on solutions of the types I(ii) and II(ii) which may well be termed *intermediate solutions* of equation (E) with $\alpha$ and $\beta$ satisfying $\alpha > \beta$.

**Theorem 2.5.** *Let $\alpha > \beta$ and suppose that* (2.1) *holds. Then, equation* (E) *possesses a solution of the type* I(ii) *if and only if*

$$\int\limits_0^\infty p(t)^{-\frac{1}{\alpha}} \bigg( \int\limits_t^\infty q(s)\,ds \bigg)^\alpha dt < \infty \ \ \textit{and} \ \int\limits_0^\infty q(t) \bigg( \int\limits_0^t p(s)^{-\frac{1}{\alpha}}\,ds \bigg)^\beta dt = \infty.$$

**Theorem 2.6.** *Let $\alpha > \beta$ and suppose that* (2.2) *holds. Then, equation (E) possesses a solution of the type type* II(ii) *if and only if*

$$\int\limits_0^\infty q(t) \bigg( \int\limits_t^\infty p(s)^{-\frac{1}{\alpha}}\,ds \bigg)^\beta < \infty \ \ \textit{and} \ \int\limits_0^\infty p(t)^{-\frac{1}{\alpha}} \bigg( \int\limits_0^t q(s)\,ds \bigg)^\alpha dt = \infty.$$

Finally, in the case where

$$\int\limits_0^\infty p(t)^{-\frac{1}{\alpha}}\,dt < \infty \ \text{ and } \int\limits_0^\infty q(t)\,dt < \infty, \tag{2.3}$$

all nonoscillatory solutions $x$ of (E) are bounded together with $Dx$ on their intervals of existence, and for any given constants $c$ and $d$, zero or non-zero, (E) has a nonoscillatory solution $x$ such that $x(\infty) = c$ and $Dx(\infty) = d$.

# References


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