Correctness and Additive Averaged Semi-Discrete Scheme for Two Nonlinear Multi-Dimensional Integro-Differential Parabolic Problems

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The note is devoted to the correctness of the initial-boundary value problems for two nonlinear multi-dimensional integro-differential equations of parabolic type. Construction and study of the additive averaged semi-discrete schemes with respect to time variable are also given. These type of equations are natural generalizations, on the one hand, of equations describing applied problems of mathematical physics and, on the other hand, of nonlinear parabolic equations in problems considered in [12] and [16]. The studied equations are based on well-known Maxwell's system arising in mathematical simulation of electromagnetic field penetration into a substance [9].

Maxwell's system is complex and its investigation and numerical resolution still yield for special cases (see, for example, [7] and references therein).

In [3] this system were proposed to the following integro-differential form

$$\frac{\partial H}{\partial t} = -\operatorname{rot}\left[a\bigg(\int\limits_0^t |\operatorname{rot} H|^2\,d\tau\bigg)\operatorname{rot} H\right], \tag{1}$$

where $a = a(S)$ is defined for $S \in [0, \infty)$.

Making certain physical assumptions in mathematical description of the above-mentioned process G. I. Laptev is constructed a new integro-differential model, which represents a generalization of the system (1)

$$\frac{\partial H}{\partial t} = a\bigg(\int\limits_\Omega \int\limits_0^t |\operatorname{rot} H|^2\;dx\,d\tau\bigg)\Delta H. \tag{2}$$

Principal characteristic peculiarities of the systems (1) and (2) are connected with the appearance in the coefficients with derivatives of higher order nonlinear terms depended on the integral of time and space variables. These circumstances requires different discussions, than it is usually necessary for the solution of local differential problems.

The literature on the questions of existence, uniqueness, and regularity of the solutions to the models of above types is very rich. In [3–7, 10, 11] and in a number of other works as well the solvability of the initial-boundary value problems for (1) type models in scalar cases are studied. The correctness of these problems in [3–7] are proved using a modified version of the Galerkin's method and compactness arguments that are used in [12, 16] for investigation the nonlinear elliptic and parabolic equations.

Let us note that the unique solvability and large time behavior of initial-boundary value problems for (2) type equations at first are given in [4].

These questions and numerical resolution of initial-boundary value problems for (1) and (2) type models are discussed in many works as well (see, for example, [7, 14, 15] and references therein).

Many authors study the Rothe's type schemes, semi-discrete schemes with space variable, finite element and finite difference approximations for a integro-differential models (see, for example, [5–8, 12, 13] and references therein).

It is very important to study decomposition analogs for the above-mentioned multi-dimensional integro-differential models as well. At present there are some effective algorithms for solving the multi-dimensional problems (see, for example, [12, 13] and references therein).

In this paper the existence and uniqueness of solutions of initial-boundary value problems are fixed. Main attention is paid to investigation of Rothe's type semi-discrete additive averaged schemes. We shall focus our attention to the (1) and (2) type multi-dimensional integro-differential scalar equations.

Let Ω is bounded domain in the n-dimensional Euclidean space R^n with sufficiently smooth boundary $\partial\Omega$. In the domain $Q=\Omega\times(0,T)$ of the variables $(x,t)=(x_1,x_2,\dots,x_n,t)$ let us consider the following equations

$$\frac{\partial U}{\partial t}-\sum_{i=1}^{n}\left[\frac{\partial}{\partial x_i}a\bigg(\int_0^t\sum_{\ell=1}^{n}\Big|\frac{\partial U}{\partial x_\ell}\Big|^2\,d\tau\bigg)\frac{\partial U}{\partial x_i}\right]=f(x,t),\ \ (x,t)\in Q, \tag{3}$$

or

$$\frac{\partial U}{\partial t}-a\bigg(\int_\Omega\int_0^t\sum_{\ell=1}^{n}\Big|\frac{\partial U}{\partial x_\ell}\Big|^2\,dx\,d\tau\bigg)\sum_{i=1}^{n}\frac{\partial^2 U}{\partial x_i^2}=f(x,t),\ \ (x,t)\in Q, \tag{4}$$

with the first type initial-boundary value homogeneous conditions

$$U(x,t)=0,\ \ (x,t)\in\partial\Omega\times[0,T], \tag{5}$$

$$U(x,0)=0,\ \ x\in\overline{\Omega}, \tag{6}$$

where T is a fixed positive constant, f is a given function of its arguments.

The problems (3), (5), (6) and (4)–(6) are similar to problems considered in [2] and [4]. Using modified version of the Galerkin's method and compactness arguments [12, 16] it is not difficult to prove the following statement.

Theorem 1. *If* $a(S)=1+S$, $f\in W_2^1(Q)$, $f(x,0)=0$, *then the problems* (3), (5), (6) *and* (4)–(6) *have solutions with the properties:*

$$U\in L_4(0,T;\overset{\circ}{W}{}_4^1(\Omega)),\quad \frac{\partial U}{\partial t}\in L_2(Q),$$

$$\sqrt{T-t}\,\frac{\partial^2 U}{\partial t\partial x_i}\in L_2(Q),\quad \sqrt{\psi}\,\frac{\partial^2 U}{\partial x_i\partial x_j}\in L_2(Q),\ \ i,j=1,\dots,n,$$

where $\psi\in C^\infty(\overline{\Omega})$, $\psi(x)>0$, *for* $x\in\Omega$; $\frac{\partial\psi}{\partial\nu}=0$, *for* $x\in\partial\Omega$ *and* ν *is the outer normal of* $\partial\Omega$.

Here we used usual L_p and W_p^k, $\overset{\circ}{W}{}_p^k$ Sobolev spaces.

The proof of the formulated theorem is divided into several steps. One of the basic step is to obtain necessary a priori estimates.

Using the scheme of investigation as in [4] it is not difficult to get the results of exponentially asymptotic behavior of solution as $t\to\infty$ of the initial-boundary value problems for the equations (3) and (4) with nonhomogeneous initial conditions.

On $[0,T]$ let us introduce a net with mesh points denoted by $t_j=j\tau$, $j=0,1,\dots,J$, with $\tau=T/J$.

Coming back to the problems (3), (5), (6) and (4)–(6) let us construct additive averaged Rothe's type schemes

$$\eta_i \frac{u_i^{j+1} - u^j}{\tau} = \frac{\partial}{\partial x_i} \left[\left(1 + \tau \sum_{k=1}^{j+1} \sum_{\ell=1}^{n} \left| \frac{\partial u_i^k}{\partial x_\ell} \right|^2 \right) \frac{\partial u_i^{j+1}}{\partial x_i} \right] + f_i^{j+1} \tag{7}$$

and

$$\eta_i \frac{u_i^{j+1} - u^j}{\tau} = \left(1 + \tau \sum_{k=1}^{j+1} \sum_{\ell=1}^{n} \int_\Omega \left| \frac{\partial u_i^k}{\partial x_\ell} \right|^2 dx \right) \frac{\partial^2 u_i^{j+1}}{\partial x_i^2} + f_i^{j+1}, \tag{8}$$

with homogeneous boundary and initial $u_i^0 = u^0 = 0$ conditions, where $u_i^j(x)$, $i = 1, \dots, n$, $j = 0, 1, \dots, J-1$ are solutions of the problems (7) and (8) consequently, and the following notations are introduced:

$$u^j(x) = \sum_{i=1}^n \eta_i u_i^j(x), \quad \sum_{i=1}^n \eta_i = 1, \ \ \eta_i > 0, \quad \sum_{i=1}^n f_i^{j+1}(x) = f^{j+1}(x) = f(x, t_{j+1}),$$

where u^j denotes approximation of exact solution U of the problem (4)–(6) at t_j. We use usual norm $\| \cdot \|$ of the space $L_2(\Omega)$.

Theorem 2. *If the problems* (3), (5), (6) *and* (4)–(6) *have sufficiently smooth solutions, then the solutions of problems* (7) *and* (8) *with homogeneous initial and boundary conditions converge to the solutions of the problems* (3), (5), (6) *and* (4)–(6) *and the following estimate is true*

$$\|U^j - u^j\| = O(\tau^{1/2}), \ \ j = 1, \dots, J.$$

Let us note that results analogical to Theorem 2 for the following variants of (3) and (4) integro-differential systems are obtained in the works [6] and [5], respectively:

$$\frac{\partial U}{\partial t} - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left[\left(1 + \int_0^t \left| \frac{\partial U}{\partial x_i} \right|^2 dx \, d\tau \right) \frac{\partial U}{\partial x_i} \right] = f(x, t),$$

and

$$\frac{\partial U}{\partial t} - \sum_{i=1}^n \left(1 + \int_\Omega \int_0^t \left| \frac{\partial U}{\partial x_i} \right|^2 dx \, d\tau \right) \frac{\partial^2 U}{\partial x_i^2} = f(x, t).$$

It is very important to construct and investigate studied in this note type models for more general type nonlinearities and for (3) and (4) type multi-dimensional systems as well.

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