The Version of Perron's Effect of Replacing the Values of Characteristic Exponents of Differential System by a Set of Positive Measure

N. A. Izobov

Department of Differential Equations, Institute of Mathematics,
National Academy of Sciences of Belarus, Minsk, Belarus
E-mail: izobov@im.bas-net.by

A. V. Il'in

Lomonosov Moscow State University, Moscow, Russia
E-mail: iline@cs.msu.su

Consider the linear differential system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^2, \quad t \geqslant t_0, \tag{1}$$

with bounded continuously differentiable on the semi-axis $[t_0, +\infty)$ coefficients and with negative characteristic exponents $\lambda_1(A) \leqslant \lambda_2(A) < 0$, which is a linear approximation for a nonlinear perturbed differential system

$$\dot{y} = A(t)y + f(t, y), \quad y \in \mathbb{R}^2, \quad t \geqslant t_0. \tag{2}$$

In this system, the so-called m-perturbation $f : [t_0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ is likewise continuously differentiable in its arguments $t \geqslant t_0$ and $y_1, y_2 \in \mathbb{R}$, of order $m > 1$ of smallness in the neighbourhood of the origin and admissible growth outside of it:

$$\|f(t, y)\| \leqslant C_f \|y\|^m, \quad C_f = const, \quad y \in \mathbb{R}^2, \quad t \geqslant t_0. \tag{3}$$

Perron's effect [17], [15, pp. 50–51; 3–11] (see also [13,14]) of replacing the values of characteristic exponents establishes the existence both of the linear system (1) with fixed characteristic exponents $\lambda_1 \leqslant \lambda_2 < 0$ and of the nonlinear system (2) with perturbation (3) of order $m = 2$ of smallness and with all infinitely extendable to the right nontrivial solutions $y(t, c)$ with initial vectors $y(t_0, c) = c = (c_1, c_2) \neq 0$. In addition, all such solutions starting at the time moment $t = t_0$ on the axis $c_1 = 0$ have exponents, equal to the higher characteristic exponent λ_2 of the initial system (1) (that allows one to consider this effect partial), and the exponents of all the rest nontrivial solutions of system (2) coincide with some $\lambda_0 > 0$ (calculated incidentally in [5, pp. 13–15]). Generalizations of that effect in various directions have been obtained in [2, 3, 6–8, 10, 11].

The question on the realization of such a (continual) version of Perron's effect, when the set $\lambda(A, f)$ of Lyapunov's exponents of all nontrivial solutions (necessarily infinitely extendable to the right) of the corresponding system (2) with perturbation (3) would have been measurable, fully belonged to the positive semi-axis, had continuum power and even positive Lebesque measure, remained open. A positive answer to this question is contained, particularly, in the theorem below which defins in a general case an explicit representation of Lyapunov's exponents of all nontrivial solutions $y(t, c)$ of the needed nonlinear system (2) through their initial values $c = (c_1, c_2) \in \mathbb{R}^2$.

Note that earlier we have constructed the perturbed differential systems (2) with an exponentially stable zero solution whose set of characteristic exponents of solutions from a sufficiently small neighbourhood of a zero solution belongs fully to the negative semi-axis and: (1) is [4] of positive measure (a positive length segment); (2) consists [18] of a countable set of nonintersecting connectivity components. Further, these sets of exponents were fully described in [1]. Obviously, the results obtained there are not connected with the realization of Perron's effect (and, all the more, with its continual analogue) of replacing negative values of characteristic exponents of the system of linear approximation (1) by positive ones for all nontrivial solutions of the nonlinear system (2) with perturbations (3) of order $m > 1$ of smallness.

The following theorem is valid [9].

Theorem. *For any parameters $m > 1$, $\lambda_1 \leqslant \lambda_2 < 0$ and bounded continuously differentiable on the axis $\mathbb{R}_0 \equiv \mathbb{R} \setminus \{0\}$ functions*

$$\psi_i : \mathbb{R}_0 \xrightarrow{on} |\beta_i, b_i| \subset [\lambda_2, +\infty), \;\; b_1 \leqslant \beta_2, \;\; i = 1, 2, \tag{4}$$

there exist the linear system (1) with characteristic exponents $\lambda_1(A) = \lambda_1 \leqslant \lambda_2 = \lambda_2(A)$ and continuously differentiable in its arguments $t \geqslant t_0$ and $y_1, y_2 \in \mathbb{R}$, the m-perturbation $f(t, y)$ such that all nontrivial solutions $y(t, c)$ of the nonlinear perturbed system (2) are infinitely extendable to the right and have characteristic exponents

$$\lambda[y(\,\cdot\,, c)] = \begin{cases} \psi_1(c_1), & c_1 \neq 0, \;\; c_2 = 0, \\ \psi_2(c_2), & c_2 \neq 0, \;\; c = (c_1, c_2) \in \mathbb{R}^2. \end{cases}$$

Corollary. *A continual version of Perron's effect of replacing the negative values of characteristic exponents of the system of linear approximation (1) by positive ones of all nontrivial solutions of the nonlinear system (2) with perturbation (3) of order $m > 1$ is realized through the functions ψ_i with sets of values, that is, by the intervals $|\beta_i, b_i| \subset (0, +\infty)$ of positive length (and of positive Lebesgue measure).*

Remark. A set of values of each of the bounded, continuously differentiable on the axis $\mathbb{R} \setminus \{0\}$ functions ψ_1 and ψ_2 may consist of two connectivity components. The corresponding analogue of the above-formulated theorem establishing also a continual version of Perron's effect holds in this case, as well. The above theorem admits generalization with replacing the inclusion $|\beta_i, b_i| \subset [\lambda_2, +\infty)$ in condition (4) by a weaker inclusion $|\beta_i, b_i| \subset [\lambda_i, +\infty)$. Moreover, by sufficiently obvious changes in the proof of theorem cited in [9], one can prove an analogous statement for the bounded functions ψ_1 and ψ_2 from the Naire first class and their Suslin sets of values.

References

[1] E. A. Barabanov and I. A. Volkov, The structure of the set of Lyapunov characteristic exponents of exponentially stable quasi-linear systems. (Russian) *Differentsial'nye Uravneniya* **30** (1994), no. 1, 3–19; translation in *Differential Equations* **30** (1994), no. 1, 1–15.

[2] A. V. Il'in and N. A. Izobov, An infinite variant of the Perron effect of change of sign of characteristic exponents of differential systems. (Russian) *Dokl. Akad. Nauk* **457** (2014), no. 2, 147–151; translation in *Dokl. Math.* **90** (2014), no. 1, 435–439.

[3] A. V. Il'in and N. A. Izobov, The Perron's effect of replacing characteristic exponents of solutions starting on a finite number of points and lines. (Russian) *Differ. Uravn.* **52** (2016), no. 8, 1139–1140.

[4] N. A. Izobov, The number of characteristic and lower exponents of an exponentially stable system with higher-order perturbations. (Russian) *Differentsial'nye Uravneniya* **24** (1988), no. 5, 784–795, 916–917; translation in *Differential Equations* **24** (1988), no. 5, 510–519.

[5] N. A. Izobov, *Lyapunov Exponents and Stability.* Cambridge Scientific Publishers, 2012.

[6] N. A. Izobov and A. V. Il'in, Finite-dimensional Perron effect of change of all values of characteristic exponents of differential systems. (Russian) *Differ. Uravn.* **49** (2013), no. 12, 1522–1536; translation in *Differ. Equ.* **49** (2013), no. 12, 1476–1489.

[7] N. A. Izobov and A. V. Il'in, Perron effect of infinite change of values of characteristic exponents in any neighborhood of the origin. (Russian) *Differ. Uravn.* **51** (2015), no. 11, 1420–1432; translation in *Differ. Equ.* **51** (2015), no. 11, 1413–1424.

[8] N. A. Izobov and A. V. Il'in, Sign reversal of the characteristic exponents of solutions of a differential system with initial data on finitely many points and lines. (Russian) *Differ. Uravn.* **52** (2016), no. 11, 1443–1456; translation in *Differ. Equ.* **52** (2016), no. 11, 1389–1402.

[9] N. A. Izobov and A. V. Il'in, Continual version of Perron's effect of replacing the values of characteristic exponents. (Russian) *Differ. Uravn.* **53** (2017), no. 11, 1427–1439.

[10] N. A. Izobov and S. K. Korovin, Multidimensional analog of the two-dimensional Perron effect of sign change of characteristic exponents for infinitely differentiable differential systems. (Russian) *Differ. Uravn.* **48** (2012), no. 11, 1466–1482; translation in *Differ. Equ.* **48** (2012), no. 11, 1444–1460.

[11] S. K. Korovin and N. A. Izobov, On the Perron sign-change effect for Lyapunov characteristic exponents of solutions of differential systems. (Russian) *Differ. Uravn.* **46** (2010), no. 10, 1388–1402; translation in *Differ. Equ.* **46** (2010), no. 10, 1395–1408.

[12] S. K. Korovin and N. A. Izobov, Realization of the Perron effect for the change of values of the characteristic exponents of solutions of differential systems. (Russian) *Differ. Uravn.* **46** (2010), no. 11, 1536–1550; translation in *Differ. Equ.* **46** (2010), no. 11, 1537–1551.

[13] V. V. Kozlov, On the mechanism of stability loss. (Russian) *Differ. Uravn.* **45** (2009), no. 4, 496–505; translation in *Differ. Equ.* **45** (2009), no. 4, 510–519.

[14] V. V. Kozlov, Stabilization of unstable equilibria by time-periodic gyroscopic forces. (Russian) *Dokl. Akad. Nauk* **429** (2009), no. 6, 762–763.

[15] G. A. Leonov, *Chaotic Dynamics and Classical Theory of Motion Stability.* (Russian) NITs RKhD, Moscow, Izhevsk, 2006.

[16] G. A. Leonov and N. V. Kuznetsov, Time-varying linearization and the Perron effects. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **17** (2007), no. 4, 1079–1107.

[17] O. Perron, Die Stabilitätsfrage bei Differentialgleichungen. (German) *M. Z.* **32** (1930), 703–728.

[18] I. A. Volkov and N. A. Izobov, Connectivity components of a set of characteristic exponents of a differential system with higher-order perturbations. (Russian) *Dokl. Akad. Nauk BSSR* **33** (1989), no. 3, 197–200.