Existence and Multiplicity of Periodic Solutions to Indefinite Singular Equations with the Phase Singularities

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The existence of a T-periodic solution to the second-order differential equation

$$u'' = h(t)g(u) \tag{1}$$

is studied in the first part. Here, $h \in L(\mathbb{R}/T\mathbb{Z})$ and $g \in C^1(\mathbb{R}_+;\mathbb{R}_+)$ ($\mathbb{R}_+$ stands for positive real numbers) is a nonincreasing function with a strong singularity at zero, i.e.,

$$\lim_{x\to 0^+} \int_x^1 g(s)\,ds = +\infty. \tag{2}$$

By a T-periodic solution to (1) we understand a T-periodic positive function $u : \mathbb{R} \to \mathbb{R}_+$ which is absolutely continuous together with its first derivative on $[0,T]$ and satisfies the equality (1) almost everywhere on $[0,T]$.

In addition to the assumptions imposed on g previously, we will need to assume the following technical condition hold:

$$\text{there exists } \ \gamma > 0 \ \text{ such that } \ \liminf_{x\to+\infty} \frac{g((1+\gamma)x)}{g(x)}\, H_- > H_+, \tag{3}$$

where

$$H_+ = \int_0^T [h(s)]_+\,ds, \quad H_- = \int_0^T [h(s)]_-\,ds,$$

denoting by $[a]_+ = \frac{1}{2}\,(|a|+a)$, $[a]_- = \frac{1}{2}\,(|a|-a)$ for any real number a. Obviously, the condition (3) implies that $\overline{h} \overset{def}{=} \frac{1}{T} \int_0^T h(s)\,ds < 0$. However, this is not restrictive because $\overline{h} < 0$ is also a necessary condition for the existence of a T-periodic solution in the case when g is strictly decreasing (see Remark 2 below). For example, the condition (3) is satisfied when $g(x) = 1/x^\lambda$ (the nonlinearity in the model equation) provided $\overline{h} < 0$.

Remark 1. Without loss of generality we can and we will assume that

$$\lim_{x\to+\infty} g(x) < 1. \tag{4}$$

Indeed, if this is not the case, we can pass to the equation

$$u'' = \widetilde{h}(t)\widetilde{g}(u),$$

where $\widetilde{h}(t) = (g_\infty + 1)h(t)$ for $t \in \mathbb{R}$, $\widetilde{g}(x) = g(x)/(g_\infty + 1)$ for $x \in \mathbb{R}_+$, and $g_\infty = \lim\limits_{x\to+\infty} g(x)$.

Theorem 1. *Let $\overline{h} < 0$, g satisfy (2), (3), and (4), and let there exist pairwise disjoint intervals $[a_k, b_k] \subset [0, T]$ $(k = 1, \dots, n)$ such that*

$$h(t) \geq 0 \;\; \text{for a.e.} \;\; t \in \bigcup_{k=1}^{n} [a_k, b_k],$$

$$h(t) \leq 0 \;\; \text{for a.e.} \;\; t \in [0, T] \setminus \bigcup_{k=1}^{n} [a_k, b_k].$$

Let, moreover, there exist $c_k \in (a_k, b_k)$ $(k = 1, \dots, n)$ such that

$$\lim_{t \to t_0^+} \int_t^{b_k} h(s) g(C_k(s - t_0))\, ds = +\infty \;\; \text{for every} \;\; t_0 \in [a_k, c_k] \quad (k = 1, \dots, n),$$

$$\lim_{t \to t_0^-} \int_{a_k}^{t} h(s) g(D_k(t_0 - s))\, ds = +\infty \;\; \text{for every} \;\; t_0 \in [c_k, b_k] \quad (k = 1, \dots, n),$$

where

$$C_k = \frac{\Gamma}{b_k - c_k} + \frac{|\overline{h}|(b_k - c_k)}{4}\,, \quad D_k = \frac{\Gamma}{c_k - a_k} + \frac{|\overline{h}|(c_k - a_k)}{4}\,,$$
$$\Gamma = g^{-1}(1) + \frac{T}{4}\, \|h\|_1.$$

Then the equation (1) has at least one T-periodic solution.

Remark 2. Note that the condition $\overline{h} < 0$ is necessary for the existence of a T-periodic solution to (1) in the case when g is a strictly decreasing function. Indeed, if u is a T-periodic solution to (1), then dividing both sides of (1) by $g(u)$ and integrating it over $[0, T]$ we arrive at

$$0 > \int_0^T \frac{u'^2(s) g'(u(s))}{g^2(u(s))}\, ds = \int_0^T \frac{u''(s)}{g(u(s))}\, ds = \int_0^T h(s)\, ds,$$

provided $h(t) \not\equiv 0$.

The equation

$$u'' = \frac{h(t)}{u^\lambda}\,, \tag{5}$$

with $\lambda > 0$, can be viewed as a particular case of (1). Thus from Theorem 1 we obtain the following assertion.

Corollary 1. *Let $\lambda \geq 1$ and let there exist pairwise disjoint intervals $[a_k, b_k] \subset [0, T]$ $(k = 1, \dots, n)$ and $\alpha > 0$ such that*

$$h(t) \geq \alpha \big[(b_k - t)(t - a_k)\big]^{\lambda - 1} \;\; \text{for a.e.} \;\; t \in [a_k, b_k] \;\; (k = 1, \dots, n),$$

$$h(t) \leq 0 \;\; \text{for a.e.} \;\; t \in [0, T] \setminus \bigcup_{k=1}^{n} [a_k, b_k].$$

Then the equation (5) has a T-periodic solution if and only if $\overline{h} < 0$.

Slightly different result can be obtained in the case where the function g possesses two singularities. Therefore, we will consider the equation of the form

$$u'' = \sigma h(t)g(u) \tag{6}$$

in the second part. Here again, $h \in L(\mathbb{R}/T\mathbb{Z})$, $\sigma > 0$ is a parameter, and $g : (A, B) \to \mathbb{R}_+$ is a continuous function with $-\infty < A < B < +\infty$. Moreover, we assume that g is continuously differentiable, and there exists $P \in (A, B)$ such that

$$g'(x) \le 0 \ \text{ for } \ x \in (A, P), \quad g'(x) \ge 0 \ \text{ for } \ x \in (P, B), \tag{7}$$

$$\lim_{x \to A^+} \int\limits_x^P g(s)\,ds = +\infty, \quad \lim_{x \to B^-} \int\limits_P^x g(s)\,ds = +\infty. \tag{8}$$

In this case, by a T-periodic solution to (6) we understand a T-periodic function $u : \mathbb{R} \to (A, B)$ which is absolutely continuous together with its first derivative on $[0, T]$ and satisfies the equality (6) almost everywhere on $[0, T]$.

Obviously, $H_+H_- \ne 0$ is a necessary condition for solvability of a periodic problem for (6).

Theorem 2. *Let* $\overline{h} \ne 0$, *g satisfy* (7), (8), *and let there exist pairwise disjoint intervals* $(a_k, b_k) \subset [0, T]$, $(x_i, y_i) \subset [0, T]$ $(k = 1, \dots, n;\ i = 1, \dots, m)$ *such that*

$$\bigcup_{k=1}^{n} [a_k, b_k] \cup \bigcup_{i=1}^{m} [x_i, y_i] = [0, T], \tag{9}$$

$$h(t) \ge 0 \ \text{ for a.e. } \ t \in \bigcup_{k=1}^{n} (a_k, b_k),$$

$$h(t) \le 0 \ \text{ for a.e. } \ t \in \bigcup_{i=1}^{m} (x_i, y_i).$$

Let, moreover, there exist $c_k \in (a_k, b_k)$ *and* $z_i \in (x_i, y_i)$ $(k = 1, \dots, n;\ i = 1, \dots, m)$ *such that*

$$\lim_{t \to t_0^+} \int\limits_t^{t_0 + \frac{P-A}{C_k}} h(s)g(A + C_k(s - t_0))\,ds = +\infty \ \text{ for every } \ t_0 \in [a_k, c_k] \ \ (k = 1, \dots, n),$$

$$\lim_{t \to t_0^-} \int\limits_{t_0 - \frac{P-A}{D_k}}^{t} h(s)g(A + D_k(t_0 - s))\,ds = +\infty \ \text{ for every } \ t_0 \in [c_k, b_k] \ \ (k = 1, \dots, n),$$

$$\lim_{t \to t_0^+} \int\limits_t^{t_0 + \frac{B-P}{K_i}} |h(s)|g(B - K_i(s - t_0))\,ds = +\infty \ \text{ for every } \ t_0 \in [x_i, z_i] \ \ (i = 1, \dots, m),$$

$$\lim_{t \to t_0^-} \int\limits_{t_0 - \frac{B-P}{L_i}}^{t} |h(s)|g(B - L_i(t_0 - s))\,ds = +\infty \ \text{ for every } \ t_0 \in [z_i, y_i] \ \ (i = 1, \dots, m).$$

where

$$C_k = \frac{B - A}{b_k - c_k}\,, \quad D_k = \frac{B - A}{c_k - a_k}\,, \quad K_i = \frac{B - A}{y_i - z_i}\,, \quad L_i = \frac{B - A}{z_i - x_i}\,.$$

Then there exists $\sigma_* > 0$ such that the equation (6) *has at least two T-periodic solutions for every $0 < \sigma < \sigma_*$ and at least one T-periodic solution for $\sigma = \sigma_*$. Moreover, there exists $\sigma^* \geq \sigma_*$ such that the equation* (6) *has no T-periodic solution for every $\sigma > \sigma^*$.*

The equation

$$u'' = \frac{\sigma h(t)}{u^\lambda (1-u)^\mu} \tag{10}$$

with $\lambda > 0$, $\mu > 0$, can be viewed as a particular case of (6). Thus from Theorem 2 we obtain the following assertion.

Corollary 2. *Let $\overline{h} \neq 0$, $\lambda \geq 1$, $\mu \geq 1$ and let there exist pairwise disjoint intervals $(a_k, b_k) \subset [0,T]$, $(x_i, y_i) \subset [0,T]$ $(k = 1, \dots, n;\ i = 1, \dots, m)$ such that* (9) *holds. Furthermore, let there exist $\alpha > 0$ such that*

$$\begin{aligned} h(t) &\geq \alpha\big[(b_k - t)(t - a_k)\big]^{\lambda-1} \ \text{ for a.e. } t \in [a_k, b_k] \ \ (k = 1, \dots, n),\\ h(t) &\leq -\alpha\big[(y_i - t)(t - x_i)\big]^{\mu-1} \ \text{ for a.e. } t \in [x_i, y_i] \ \ (i = 1, \dots, m). \end{aligned}$$

Then there exists $\sigma_* > 0$ such that the equation (10) *has at least two T-periodic solutions for every $0 < \sigma < \sigma_*$ and at least one T-periodic solution for $\sigma = \sigma_*$. Moreover, there exists $\sigma^* \geq \sigma_*$ such that the equation* (10) *has no T-periodic solution for every $\sigma > \sigma^*$.*

The proofs of the above-presented results can be found in the papers [1, 2].

References

[1] R. Hakl and M. Zamora, Periodic solutions to second-order indefinite singular equations. *J. Differential Equations* **263** (2017), no. 1, 451–469.

[2] R. Hakl, M. Zamora, Existence and multiplicity of periodic solutions to indefinite singular equations having a non-monotone term with two singularities. (submitted).