Global Bifurcation of a Unique Limit Cycle in Some Class of Planar Systems

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1 Introduction

We consider the planar autonomous differential systems

$$\frac{dx}{dt} = P(x,y,\lambda), \quad \frac{dy}{dt} = Q(x,y,\lambda) \tag{1.1}$$

depending on a scalar parameter $\lambda \in \mathbb{R}$. Our goal is to derive conditions on P and Q such that there is a $\lambda_0 \in \mathbb{R}$ with the property that for all $\lambda > \lambda_0$ system (1.1) has a unique limit cycle in the phase plane which is hyperbolic and stable. Our approach to treat this problem is based on the bifurcation theory of planar autonomous systems. The underlying idea of our approach can be formulated as follows: We assume that $\lambda = \lambda_0$ and $\lambda = +\infty$ are bifurcation points of system (1.1) connected with the appearance of a limit cycle which is hyperbolic and stable, and we suppose that the interval $(\lambda_0, +\infty)$ does not contain any bifurcation point of system (1.1). The class of Dulac–Cherkas functions, the theory of one-parameter families of rotated vector fields and singularly perturbed systems are key ingredients in our approach [1, 3–5, 8–10]. In the Appendix their basic properties are summarized. We illustrate our approach by an example.

2 Assumptions. Main result

Consider system (1.1) under the following assumptions:

(A_1) *$P, Q : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are sufficiently smooth.*

(A_2) *System* (1.1) *has $\forall\, \lambda \in \mathbb{R}$ a unique equilibrium $E(\lambda)$ in the finite part of the phase plane.*

Without loss of generality we may suppose that $E(\lambda)$ is located at the origin $\forall\, \lambda$.

(A_3) *The origin changes its stability at $\lambda = \lambda_0$ and is unstable for $\lambda > \lambda_0$.*

(A_4) *There exists for $\lambda > \lambda_0$ a Dulac–Cherkas function $\Psi(x,y,\lambda)$ of system* (1.1) *in the phase plane such that the set $\mathcal{W}_\lambda := \{(x,y) \in \mathbb{R}^2 :\ \Psi(x,y,\lambda) = 0\}$ consists of a unique oval surrounding the origin.*

(A_5) *For $\lambda > \lambda_0$ there is a one-to-one mapping*

$$\overline{x} = \varphi_1(x, y, \lambda), \quad \overline{y} = \psi_1(x, y, \lambda)$$

such that system (1.1) *will be transformed into the system*

$$\frac{d\overline{x}}{dt} = \overline{P}(\overline{x}, \overline{y}, \lambda), \quad \frac{d\overline{y}}{dt} = \overline{Q}(\overline{x}, \overline{y}, \lambda) \tag{2.1}$$

with the following properties:

(i) *The functions $\overline{P}$ and $\overline{Q}$ have for $\lambda > \lambda_0$ the same smoothness as the functions P and Q.*

(ii) *The origin is the unique equilibrium of system* (2.1) $\forall\, \lambda > \lambda_0$.

(iii) λ_0 *is a Hopf bifurcation point for system* (2.1) *connected with the bifurcation of a stable limit cycle $\overline{\Gamma}_\lambda$ from the origin for increasing λ which is positively* (*that is anti-clockwise*) *oriented.*

(iv) *System* (2.1) *represents for $\lambda > \lambda_0$ a one-parameter family of positively rotated vector fields.*

(A_6) *For $\lambda > \lambda_0$ there is a one-to-one mapping*

$$\widetilde{x} = \varphi_2(x, y, \lambda), \quad \widetilde{y} = \psi_2(x, y, \lambda), \quad \tau = \chi(t, \lambda),$$

where τ increases with t for any $\lambda > \lambda_0$, such that system (1.1) *will be transformed into the system*

$$\frac{d\widetilde{x}}{d\tau} = \widetilde{P}(\widetilde{x}, \widetilde{y}, \varepsilon), \quad \varepsilon\,\frac{d\widetilde{y}}{d\tau} = \widetilde{Q}(\widetilde{x}, \widetilde{y}, \varepsilon) \tag{2.2}$$

with the following properties:

(i) *There is a smooth function $\zeta : (\lambda_0, +\infty) \to \mathbb{R}^+$ with $\zeta(\lambda) \to 0$ as $\lambda \to +\infty$ such that $\varepsilon = \zeta(\lambda)$.*

(ii) *The functions $\widetilde{P}$ and $\widetilde{Q}$ have for $\varepsilon > 0$ the same smoothness as the functions P and Q.*

(iii) *There is a sufficiently small positive number δ such that for $\varepsilon \in (0, \delta)$ system* (2.2) *has a family $\{\widetilde{\Gamma}_\varepsilon\}$ of uniformly bounded hyperbolic stable limit cycles which surround the origin and are positively oriented.*

The following theorem is our main result.

Theorem 2.1. *Under the assumptions (A_1)–(A_6) system* (1.1) *has for $\lambda > \lambda_0$ a unique family $\{\Gamma_\lambda\}$ of limit cycles which are hyperbolic, stable and positively oriented, and whose amplitudes are bounded on any bounded λ-interval.*

3 Example

We present an application of Theorem 2.1 for the Liénard system

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x - \lambda(x^{2q} - 1)y \tag{3.1}$$

with $q \in \mathbb{N}$. For $q = 1$, system (3.1) represents the famous van der Pol system. We show that system (3.1) has the same properties as the van der Pol system. For this purpose we prove that the assumptions (A_1)–(A_6) are fulfilled for system (3.1). In particular, we get the following results

Lemma 3.1. *The function*

$$\Psi(x,y,\lambda) \equiv x^2 + y^2 - 1$$

is a Dulac–Cherkas function for system (3.1) *in the phase plane for* $\lambda > 0$.

Finally, we apply Theorem 2.1 and get the result

Theorem 3.2. *System* (3.1) *has for all* $\lambda > 0$ ($\lambda < 0$) *a unique limit cycle which is hyperbolic stable* (*unstable*) *and positively oriented.*

Full version of the derived results as a corresponding paper has been submitted for publication.

4 Appendix

Suppose that P, Q satisfy assumption (A_1). We denote by $X(\lambda)$ the vector field defined by system (1.1), by Λ some λ-interval and by Ω some region in $\mathbb{R}^2$.

Definition 4.1. A function $\Psi : \Omega \times \Lambda \to \mathbb{R}$ with the same smoothness as P, Q is called a Dulac–Cherkas function of system (1.1) in Ω for $\lambda \in \Lambda$ if there exists a real number $\kappa \neq 0$ such that

$$\Phi := (\operatorname{grad}\Psi, X(\lambda)) + \kappa\Psi \operatorname{div} X(\lambda) > 0 \ \ (< 0) \quad \text{for} \ \ (x,y,\lambda) \in \Omega \times \Lambda. \tag{4.1}$$

Remark 4.2. Condition (4.1) can be relaxed by assuming that Φ may vanish in Ω on a set of measure zero, and that no closed curve of this set is a limit cycle of (1.1).

The following two theorems can be found in [2].

Theorem 4.3. *Let* Ψ *be a Dulac–Cherkas function of* (1.1) *in* Ω *for* $\lambda \in \Lambda$. *Then any limit cycle* Γ_λ *of* (1.1) *in* Ω *is hyperbolic and its stability is determined by the sign of the expression* $\kappa\Phi\Psi$ *on* Γ_λ.

Theorem 4.4. *Let* Ω *be a* p*-connected region, let* Ψ *be a Dulac–Cherkas function of* (1.1) *in* Ω *such that the set* $\mathcal{W}_\lambda := \{(x,y) \in \Omega : \ \Psi(x,y,\lambda) = 0\}$ *consists of* s *ovals in* Ω. *Then system* (1.1) *has at most* $p - 1 + s$ *limit cycles in* Ω.

The following facts can be found in [7].

Definition 4.5. Let the assumption (A_1) be satisfied. System (1.1) is said to define a one-parameter family of negatively (positively) rotated vector fields for $\lambda \in \Lambda$ if for $\lambda \in \Lambda$ the equilibria of system (1.1) are isolated and at all ordinary points it holds

$$\Delta(x,y,\lambda) := P(x,y,\lambda)\,\frac{\partial Q(x,y,\lambda)}{\partial\lambda} - Q(x,y,\lambda)\,\frac{\partial P(x,y,\lambda)}{\partial\lambda} < 0 \ \ (> 0).$$

Remark 4.6. This condition can be relaxed by assuming that Δ vanishes on a set of measure zero and that no closed curve of this set is a limit cycle of (1.1).

Theorem 4.7. *Suppose that the assumptions* (A_1) *and* (A_2) *are satisfied and that system* (1.1) *represents a one-parameter family of negatively* (*positively*) *rotated vector fields. Let* $\{\Gamma_\lambda\}$ *be a family of hyperbolic stable limit cycles of system* (1.1) *with positive orientation. Then the amplitude of* Γ_λ *decreases monotonically with decreasing* (*increasing*) λ, *and the family terminates at* $\lambda = \lambda_*$ *when* Γ_{λ_*} *represents an equilibrium.*

Consider the singularly perturbed system

$$\frac{dx}{dt} = f(x,y), \quad \varepsilon\,\frac{dy}{dt} = g(x,y) \tag{4.2}$$

under the following assumptions

(C_1) *$f, g : \mathbb{R}^2 \to \mathbb{R}$ are sufficiently smooth, ε is a small positive parameter.*

(C_2) *The origin is the unique equilibrium of system* (4.2) *in the finite part of the phase plane. It is unstable for $\varepsilon > 0$. The trajectories are positively oriented near the origin.*

(C_3) *$g(x,y) = 0$ has the unique simple solution $x = \varphi(y)$, where φ is sufficiently smooth and satisfies*

$$\varphi(0) = 0, \quad \varphi'(0) < 0.$$

$\varphi'(y) = 0$ has exactly two real roots y_- and y_+ satisfying

$$y_- < 0, \ \ \varphi''(y_-) < 0, \ \ y_+ > 0, \ \ \varphi''(y_+) > 0.$$

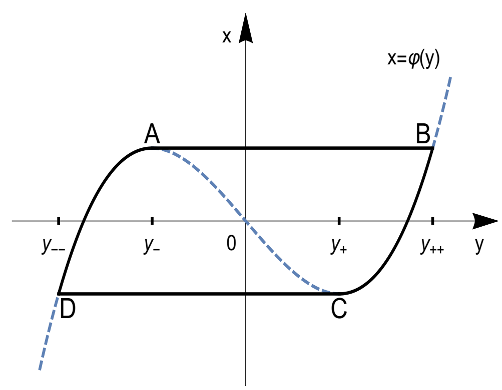

Figure 1. Closed curve $\mathcal{Z}_0$.

Using assumption (C_3) we can define a closed curve $\mathcal{Z}_0$ in the phase plane consisting of two finite segments of the curve $x = \varphi(y)$ bounded by the points $D = (y_-, \varphi(y_+))$, $A = (y_-, \varphi(y_-))$ and $C = (y_+, \varphi(y_+))$, $B = (y_{++}, \varphi(y_-))$ and of two finite segments of the straight lines $x = \varphi(y_-)$ and $x = \varphi(y_+)$ bounded by the points A, B and D, C, respectively (see Figure 1).

The following theorem is a special case of a more general theorem by E. F. Mishchenko and N. Kh. Rozov in [6].

Theorem 4.8. *Under the assumptions (C_1)–(C_3), system* (4.2) *has for sufficiently small ε a unique limit cycle Γ_ε in a small neighborhood of $\mathcal{Z}_0$ which is stable and positively oriented*

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