On Relations Between Regularity Coefficients of Linear Difference Equations

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Denote by $\mathcal{M}d_s$ the class of difference equations

$$x(n+1)=A(n)x(n),\ \ x(n)\in\mathbb{R}^s,\ \ n\in\mathbb{N}_0\stackrel{\text{def}}{=}\mathbb{N}\cup\{0\},\tag{1}$$

of dimension $s\geq 2$ with matrix coefficient $A(\,\cdot\,)\colon\mathbb{N}_0\to\operatorname{End}\mathbb{R}^s$ such that

$$\sup\Big\{\max\big\{\|A(n)\|,\|A^{-1}(n)\|\big\}:\ n\in\mathbb{N}_0\Big\}<+\infty,$$

where $\|\cdot\|$ is the operator norm generated by the Euclidean norm in $\mathbb{R}^s$ (the Euclidean norm will be denoted by the same symbol). In our further consideration we will identify the system (1) with its coefficient matrix and we will write $A(\,\cdot\,)\in\mathcal{M}d_s$, or simply $A\in\mathcal{M}d_s$. The solution $x(\,\cdot\,)$ of the system (1) is a sequence $x(\,\cdot\,)=(x(n))_{n=0}^{+\infty}$ of vectors from $\mathbb{R}^s$ satisfying for all $n\in\mathbb{N}_0$ the equation (1). The set of all solution of a system $A(\,\cdot\,)\in\mathcal{M}d_s$ with standard operations of multiplication by scalars and vector addition forms a linear space over $\mathbb{R}$, which will be denoted by $\mathcal{X}_A$. A natural isomorphism between linear spaces $\mathbb{R}^s$ and $\mathcal{X}_A$ is given by a bijection $\xi\longleftrightarrow x(\,\cdot\,;\xi)$. For natural numbers $k\geq m$ denote by $\mathcal{A}(k,m)$ a matrix equal to

$$\prod_{i=k-1}^{m}A(i)\ \text{ if }\ k>m$$

and to identity $s\times s$ matrix I_s if $k=m$. With this notation we have $x(n;x_0)=\mathcal{A}(n,0)x_0$ and $\mathcal{A}(k,m)=\Phi(k)\Phi(m)^{-1}$, where $\Phi(\,\cdot\,)$ is any fundamental matrix of the system (1).

Together with the system (1) we will consider the adjoint system

$$y(n+1)=A^{-T}(n)y(n),\ \ y(n)\in\mathbb{R}^s,\ \ n\in\mathbb{N}_0,\ \ A^{-T}(\,\cdot\,)\stackrel{\text{def}}{=}(A^T(\,\cdot\,))^{-1}.\tag{2}$$

It is obvious that the system adjoint to the system (2) is the system (1), therefore the systems (1) and (2) are called mutually adjoint.

With each system $A\in\mathcal{M}d_s$ we associate the so-called: Lyapunov regularity coefficients $\sigma_L(A)$, Perron regularity coefficient $\sigma_P(A)$ and Grobman regularity coefficient $\sigma_G(A)$ [3, 4, 6, 8]. The role of these coefficients lies in the fact that they essentially characterize the response of the system (1) to linear exponentially decreasing and non-linear of higher order of smallness perturbations. In particular, the equality of at least one of them (and then any) to zero is equivalent to the regularity in Lyapunov sense of the system (1).

Now, we will present definitions of regularity coefficients of a system $A \in \mathcal{M}d_s$. Let $\lambda_1(A) \le \cdots \le \lambda_s(A)$ denote the Lyapunov exponents of the system (1) and $\mu_1(A) \ge \cdots \ge \mu_s(A)$ the Lyapunov exponents of the adjoint system (2) (the first ones are numbered in non-decreasing order and the second in non-increasing order). By $\Psi(A)$ we denote the set of all fundamental matrices of the system (1). For any sequence $(X(n))_{n=0}^{+\infty}$ of $s \times s$ matrices by $\lambda_i[X]$ we denote the Lyapunov exponent of its i-th column, $i = 1, \dots, s$. The Lyapunov, Perron and Grobman regularity coefficients are given by the following formulae:

$$\sigma_L(A) \stackrel{\text{def}}{=} \sum_{i=1}^{s} \lambda_i(A) - \varliminf_{n\to+\infty} n^{-1} \ln |\det \mathcal{A}(n,0)|, \tag{3}$$

$$\sigma_P(A) \stackrel{\text{def}}{=} \max_{1\le i\le s} \big\{\lambda_i(A) + \mu_i(A)\big\}, \tag{4}$$

$$\sigma_G(A) \stackrel{\text{def}}{=} \inf_{\Phi\in\Psi(A)} \max_{1\le i\le s} \big\{\lambda_i[\Phi] + \lambda_i[\Phi^{-T}]\big\}. \tag{5}$$

Let us notice that by the formula (3) the Lyapunov regularity coefficient of the adjoint system (2) is given by

$$\sigma_L(A^{-T}) = \sum_{i=1}^{s} \mu_i(A) + \varlimsup_{n\to+\infty} n^{-1} \ln |\det \mathcal{A}(n,0)|.$$

For continuous time system, in the monograph [2, pp. 55, 74], it has been shown that the regularity coefficients of any system $A \in \mathcal{M}_s$, $s \ge 2$ satisfy the following relations

$$0 \le \sigma_P(A) \le \sigma_G(A) \le s\sigma_P(A) \ \text{ and } \ 0 \le \sigma_G(A) \le \sigma_L(A) \le s\sigma_G(A),$$

where $\mathcal{M}_s$ denotes the set of liner differential systems with piecewise continuous coefficient $s \times s$-matrices uniformly bounded on the nonnegative half-line $[0,+\infty)$. In addition, in [2, p. 151] it has been shown that all these inequalities are achievable and there exists a system $A \in \mathcal{M}_s$ such that the Lyapunov, Perron and Grobman regularity coefficients are pairwise different. In the monograph [5, pp. 21, 22] the following improvement of the last inequality

$$0 \le \sigma_P(A) \le \sigma_G(A) \le \sigma_L(A) \le s\sigma_P(A) \tag{6}$$

has been proved for any system $A \in \mathcal{M}_s$, $s \ge 2$.

In the paper [9], it has been shown that the inequalities (6) describe all possible relations between the regularity coefficients of differential systems. In other words, it was shown that for any natural $s \ge 2$ and ordered triple of numbers (p, g, l) satisfying the inequalities $0 \le p \le g \le \ell \le sp$, there exists a system $A \in \mathcal{M}_s$, such that $\sigma_P(A) = p$, $\sigma_G(A) = g$ and $\sigma_L(A) = \ell$. For the difference systems an analogical result was presented in [1].

From the definitions (4) and (5) of the Perron $\sigma_P(A)$ and Grobman $\sigma_G(A)$ regularity coefficients it follows that $\sigma_P(A) = \sigma_P(A^{-T})$ and $\sigma_G(A) = \sigma_G(A^{-T})$. However, analogical equality for Lyapunov $\sigma_L(A)$ regularity coefficient does not hold in general. The example of systems $A \in \mathcal{M}_s$, such that $\sigma_L(A) \ne \sigma_L(A^{-T})$, is constructed in [2, p. 155]. Analogical example of system $A \in \mathcal{M}d_s$ can be constructed. The question about the relation between the Lyapunov regularity coefficients of mutually adjoint systems, i.e. the question about description of the set of pairs $(\sigma_L(A), \sigma_L(A^{-T}))$ was solved in [9]. In this paper it has been shown that for each natural $s \ge 2$ and each non-negative numbers ℓ and ℓ^* there exists a system $A \in \mathcal{M}_s$ such that $\ell = \sigma_L(A)$ and $\ell^* = \sigma_L(A^{-T})$ if and only if $s^{-1}\ell^* \le \ell \le s\ell^*$. Analogous result for the discrete-time systems has been proved in [7].

From (6) it is straightforward to obtain the following chain of the inequalities

$$0 \le \sigma_P(A) \le \sigma_G(A) \le \min\big(\sigma_L(A), \sigma_L(-A^T)\big) \le \max\big(\sigma_L(A), \sigma_L(-A^T)\big) \le s\sigma_P(A).$$

Consequently, it is important to know for which nonnegative numbers p, g, ℓ, and ℓ^* one can construct a system $A \in \mathcal{M}d_s$ satisfying the equalities $\sigma_p(A) = p$, $\sigma_G(A) = g$, $\sigma_L(A) = \ell$, and $\sigma_L(A^{-T}) = \ell^*$. The answer to the last question is given by the following theorem.

Theorem. *For any integer $s \geq 2$ and an ordered quadruple (p, g, ℓ, ℓ^*) of real numbers satisfying the inequalities $0 \leq p \leq g \leq \min(\ell, \ell^*) \leq \max(\ell, \ell^*) \leq sp$, there exists a system $A \in \mathcal{M}d_s$ such that $\sigma_P(A) = p$, $\sigma_G(A) = g$, $\sigma_L(A) = \ell$, and $\sigma_L(A^{-T}) = \ell^*$.*

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