On the Solvability of a Boundary Value Problem for Fourth Order Linear Functional Differential Equations

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Motivated by article [7], in this paper, we consider a boundary value problem for functional differential equation of the fourth order. We obtain sharp sufficient conditions for the existence and uniqueness of solutions.

Boundary value problems for fourth order functional differential equations are considered in [2–6, 8].

Definition 1. A linear operator T from the space of all continuous real functions $\mathbf{C}[0,1]$ into the space of all integrable functions $\mathbf{L}[0,1]$ is called positive if it maps every nonnegative continuous function into an almost everywhere nonnegative integrable function.

Consider the boundary value problem for a fourth order functional differential equation:

$$\begin{cases} x^{(4)}(t) = -(Tx)(t) + f(t), \ \ t \in [0,1], \\ x(0) = c_1, \quad \dot{x}(0) = c_2, \quad x(1) = c_3, \quad \dot{x}(1) = c_4, \end{cases} \tag{1}$$

where $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ is a linear bounded operator, $f \in \mathbf{L}[0,1]$, c_i, $i = 1, \dots, 4$, are real constants.

This problem possesses the Fredholm property (see, for example, [7]). Therefore, this problem is uniquely solvable if and only if the homogeneous problem

$$\begin{cases} x^{(4)}(t) = -(Tx)(t), \ \ t \in [0,1], \\ x(0) = 0, \quad \dot{x}(0) = 0, \quad x(1) = 0, \quad \dot{x}(1) = 0, \end{cases} \tag{2}$$

has only the trivial solution.

The Green function $G(t,s)$ of problem (2) is defined by the equality

$$G(t,s) = \begin{cases} \dfrac{t^2(1-s)^2(3s-t-2st)}{6} & \text{if } \ 0 \le t \le s \le 1, \\[2mm] \dfrac{(1-t)^2 s^2(3t-s-2st)}{6} & \text{if } \ 0 \le s < t \le 1. \end{cases}$$

So, problem (2) is equivalent to the equation

$$x = -GTx,$$

where $(Gz)(t) = \int\limits_0^1 G(t,s)z(s)\,ds$, $t \in [0,1]$, is the Green operator.

By using the principle of contraction mappings, we get that problem (1) has a unique solution if at least one from the following inequalities is fulfilled:

$$\|T\|_{\mathbf{C}[0,1]\to\mathbf{L}[0,1]} < 192, \quad \|T\|_{\mathbf{C}[0,1]\to\mathbf{L}_\infty[0,1]} < 384.$$

Let $p \in \mathbf{L}[0,1]$ be non-negative function.

Definition 2. $\mathbb{S}(p)$ is a set of all linear positive operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ satisfying the condition
$$T\mathbf{1} = p,$$
where $\mathbf{1}$ is the unit function.

Theorem 1. *Boundary value problem* (1) *has a unique solution for every operator $T \in \mathbb{S}(p)$ if and only if the inequality*
$$\begin{vmatrix} 1 + \int\limits_{t_0}^{1} G(t_1,s)p(s)\,ds & 1 + \int\limits_{0}^{1} G(t_1,s)p(s)\,ds \\ \int\limits_{t_0}^{1} G(t_2,s)p(s)\,ds & 1 + \int\limits_{0}^{1} G(t_2,s)p(s)\,ds \end{vmatrix} > 0$$
holds for all $0 \le t_1 \le t_2 \le 1$ and all $t_0 \in [0,1]$.

The base of the proof of Theorem 1 is the following lemma.

Lemma 1. *Let $p \in \mathbf{L}[0,1]$ be a non-negative function. Then the boundary value problem* (1) *has a unique solution for every operator $T \in \mathbb{S}(p)$ if and only if the problem*
$$\begin{cases} x^{(4)}(t) = -p_1(t)x(t_1) - p_2(t)x(t_2), & t \in [0,1], \\ x(0) = 0, \quad \dot{x}(0) = 0, \quad x(1) = 0, \quad \dot{x}(1) = 0, \end{cases}$$
has only the trivial solution for all $0 \le t_1 \le t_2 \le 1$ and for all functions p_1, $p_2 \in \mathbf{L}[0,1]$ such that
$$p_1(t) + p_2(t) = p, \ \ 0 \le p_i(t) \le p(t), \ \ t \in [0,1], \ \ i = 1,2.$$

Consider the case where $p(t) \equiv P > 0$ is a constant.

Lemma 2. *Let $p(t) \equiv P > 0$ be a constant. If for some $T \in \mathbb{S}(P)$ problem* (2) *has a non-trivial solution, then for some $T \in \mathbb{S}(P)$ problem* (2) *has a symmetric non-trivial solution x such that $x(t) = -x(1-t)$ for all $t \in [0,1]$.*

By Lemma 2, we can put $t_0 = 1/2$ and $t_2 = 1 - t_1$, $t_1 \in [0,1/2]$ in Theorem 2 if $p(t) \equiv P$. So, by Theorem 1, in this case problem (1) is uniquely solvable for all operators $T \in \mathbb{S}(P)$ if and only if
$$P < \frac{1}{\max\limits_{t_1 \in [0,1/2]} \Big(\int\limits_{1/2}^{1} G(1-t_1,s)\,ds - \int\limits_{1/2}^{1} G(t_1,s)\,ds \Big)} = \frac{192}{\max\limits_{t1 \in [0,1/2]} t_1^2(1-2t_1)(3-4t_1)} = \frac{1760\sqrt{33}}{3} - 416 \approx 2954.$$

Corollary 1. *Let $p \in \mathbf{L}[0,1]$ be a non-negative function such that*
$$\operatorname*{vrai\,sup}_{t \in [0,1]} p(t) \le \frac{1760\sqrt{33}}{3} - 416, \quad p(t) \not\equiv \frac{1760\sqrt{33}}{3} - 416.$$
Then boundary value problem (1) *is uniquely solvable for all operators $T \in \mathbb{S}(p)$.*

The constant in the Corollary 1 is sharp.

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