Asymptotic Properties of Special Classes of Solutions of Second-Order Differential Equations with Nonlinearities in Some Sense Near to Regularly Varying

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The following differential equation is considered in the work

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y')\exp\big(R(|\ln|yy'||)\big). \tag{1}$$

Here $\alpha_0 \in \{-1,1\}$, $p : [a;\omega[\to]0;+\infty[$ $(-\infty < a < \omega \le +\infty)$, $\varphi_i : \Delta_{Y_i} \to]0;+\infty[$ are continuous functions, $Y_i \in \{0,\pm\infty\}$ $(i=0,1)$, Δ_{Y_i} is a one-sided neighborhood of Y_i, every function $\varphi_i(z)$ $(i=0,1)$ is a regularly varying function as $z \to Y_i$ $(z \in \Delta_{Y_i})$ of order σ_i, $\sigma_0+\sigma_1 \ne 1$, $\sigma_1 \ne 0$, the function $R :]0;+\infty[\to]0;+\infty[$ is continuously differentiable and regularly varying on infinity of the order μ, $0<\mu<1$, the derivative function of the function R is monotone.

Definition. A solution y of equation (1) is called $P_\omega(Y_0,Y_1,\lambda_0)$-solution if it is defined on $[t_0,\omega[\subset [a,\omega[$ and

$$\lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i=0,1), \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y(t)y''(t)} = \lambda_0.$$

A lot of works (see, for example, [2, 3]) have been devoted to the establishing asymptotic representations of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equations of the form (1), in which $R \equiv 0$. The $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation (1) are regularly varying functions as $t\uparrow\omega$ of index $\frac{\lambda_0}{\lambda_0-1}$ if $\lambda_0 \in R\setminus\{0,1t\}$. The asymptotic properties and necessary and sufficient conditions of existence of such solutions of equation (1) have been obtained in [1].

The cases $\lambda_0 \in \{0,1\}$ and $\lambda_0=\infty$ are special. $P_\omega(Y_0,Y_1,1)$-solutions of equation (1) are rapidly varying functions as $t\uparrow\omega$. The cases $\lambda_0=0$ and $\lambda_0=\infty$ are cases of the most difficulty because in this cases such solutions or their derivatives are slowly varying functions as $t\uparrow\omega$. Some results about asymptotic properties and existence of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation (1) in these special cases are presented in the work.

We say that a slowly varying as $z\to Y$ $(z\in\Delta_Y)$ function $\theta : \Delta_Y \to]0;+\infty[$ satisfies the condition S if for any continuous differentiable function $L : \Delta_{Y_i} \to]0;+\infty[$ such that

$$\lim_{\substack{z\to Y_i\\ z\in\Delta_{Y_i}}} \frac{zL'(z)}{L(z)} = 0,$$

the following equality

$$\Theta(zL(z)) = \Theta(z)(1+o(1)) \ \text{ is true as } \ z\to Y \ \ (z\in\Delta_Y).$$

Let us introduce the following notations.

$$\pi_\omega(t) = \begin{cases} t & \text{if } \ \omega=+\infty,\\ t-\omega & \text{if } \ \omega<+\infty, \end{cases} \qquad \Theta_i(z) = \varphi_i(z)|z|^{-\sigma_i} \ \ (i=0,1),$$

$$I(t)=\alpha_0\int\limits_{A_\omega}^{t}p(\tau)\,d\tau,\qquad A_\omega=\begin{cases}a & \text{if } \int\limits_a^\omega p(\tau)\,d\tau=+\infty,\\ \omega & \text{if } \int\limits_a^\omega p(\tau)\,d\tau<+\infty.\end{cases}$$

In case $\lim\limits_{t\uparrow\omega}\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}=Y_1$, we put

$$J(t)=\int\limits_{B_\omega}^{t}\Big|I(\tau)\Theta_1\Big(\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau,$$

$$B_\omega=\begin{cases}b_1 & \text{if } \int\limits_{b_1}^{\omega}\Big|I(\tau)\Theta_1\Big(\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau=+\infty,\\ \omega & \text{if } \int\limits_{b_1}^{\omega}\Big|I(\tau)\Theta_1\Big(\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau<+\infty,\end{cases}$$

$$N_1(t)=\frac{(1-\sigma_1)I(t)\big|(1-\sigma_1)I(t)\Theta_1\big(\frac{y_1^0}{|\pi_\omega(t)|}\big)\big|^{\frac{1}{\sigma_1-1}}}{I'(t)R'(|\ln|\pi_\omega(t)||)}\,,$$

and in case $\lim\limits_{t\uparrow\omega}|\pi_\omega(\tau)|\operatorname{sign} y_0^0=Y_0$, we put

$$I_0(t)=\alpha_0\int\limits_{A_\omega^0}^{t}p(\tau)|\pi_\omega(\tau)|^{\sigma_0}\Theta_0\big(|\pi_\omega(\tau)|\operatorname{sign} y_0^0\big)\,d\tau,$$

$$A_\omega^0=\begin{cases}b_2 & \text{if } \int\limits_{b_2}^{\omega}p(t)|\pi_\omega(t)|^{\sigma_0}\Theta_0\big(|\pi_\omega(t)|\operatorname{sign} y_0^0\big)\,dt=+\infty,\\ \omega & \text{if } \int\limits_{b_2}^{\omega}p(t)|\pi_\omega(t)|^{\sigma_0}\Theta_0\big(|\pi_\omega(t)|y_0^0\big)\,dt<+\infty,\end{cases}$$

$$N_2(t)=\alpha_0p(t)|\pi_\omega(t)|^{\sigma_0+1}\Theta_0\big(|\pi_\omega(t)|\operatorname{sign} y_0^0\big).$$

Here $b_1,b_2\in[a;\omega[$ are chosen in such a way that $\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}\in\Delta_{Y_1}$ as $t\in[b_1;\omega]$ and $|\pi_\omega(\tau)|\operatorname{sign} y_0^0\in\Delta_{Y_0}$ as $t\in[b_2;\omega]$.

The first two theorems are devoted to the existence $P_\omega(Y_0,Y_1,0)$-solutions of equation (1). Such solutions are slowly varying functions as $t\uparrow\omega$, that makes difficulties in their investigations.

Theorem 1. *Let in equation* (1) *the function φ_1 satisfy the condition S and the following condition take place*

$$\lim_{t\uparrow\omega}\frac{R(|\ln|\pi_\omega(t)||)J(t)}{\pi_\omega(t)\ln|\pi_\omega(t)|J'(t)}=0. \tag{2}$$

Then for the existence of $P_\omega(Y_0,Y_1,0)$-solutions of equation (1) *the following conditions are necessary and sufficient*

$$\lim_{t\uparrow\omega}y_0^0|J(t)|^{\frac{1-\sigma_1}{1-\sigma_0-\sigma_1}}=Y_0,\quad \lim_{t\uparrow\omega}\frac{J'(t)}{y_1^0|J(t)|}=Y_1,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)I'(t)}{I(t)}=\sigma_1-1,$$

$$\frac{I(t)}{y_1^0(1-\sigma_1)} > 0, \quad \frac{y_0^0 y_1^0(1-\sigma_1)J(t)}{1-\sigma_0-\sigma_1} > 0 \ \ as \ \ t \in [b_1, \omega[\,.$$

For such solutions the following asymptotic representations take place as $t \uparrow \omega$

$$\frac{y(t)}{\big|\exp(R(|\ln|y(t)y'(t)||))\varphi_0(y(t))\big|^{\frac{1}{1-\sigma_1}}} = \frac{1-\sigma_0-\sigma_1}{1-\sigma_1}\,|1-\sigma_1|^{\frac{1}{1-\sigma_1}} J(t)[1+o(1)],$$
$$\frac{y'(t)}{y(t)} = \frac{(1-\sigma_1)J'(t)}{(1-\sigma_0-\sigma_1)J(t)}\,[1+o(1)].$$

Theorem 2. *Let condition* (2) *of Theorem* 1 *be not satisfied,* p *be a twice continuously differentiable function, function* φ_1 *satisfy the condition* S *and the following condition take place*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)N_1'(t)}{R'(|\ln|\pi_\omega(t)||)N_1(t)} = 0.$$

For the existence of such $P_\omega(Y_0, Y_1, 0)$*-solutions of equation* (1)*, that finite or infinite limit* $\lim\limits_{t\uparrow\omega} \frac{\pi_\omega(t)y''(t)}{y'(t)}$ *exists, the following conditions are necessary and sufficient*

$$\lim_{t\uparrow\omega} y_0^0\Big(\exp\big(R(|\ln|\pi_\omega(t)||)\big)^{\frac{\sigma_1-1}{1-\sigma_0-\sigma_1}}\Big) = Y_0, \quad \lim_{t\uparrow\omega} \frac{-\alpha_0}{\pi_\omega(t)} = Y_1, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)I'(t)}{I(t)} = \frac{\sigma_1-1}{\alpha_0}\,,$$
$$\alpha_0 y_1^0 \pi_\omega(t) < 0, \quad \alpha_0(1-\sigma_1)(1-\sigma_0-\sigma_1)y_0^0 R'\big(|\ln|\pi_\omega(t)||\big) > 0 \ \ as \ \ t \in [a, \omega[\,.$$

For such solutions the following asymptotic representations take place as $t \uparrow \omega$

$$\frac{y(t)}{|\varphi_0(y(t))\exp(R(|\ln|y(t)y'(t)||))\big|^{\frac{1}{1-\sigma_1}}} = (1-\sigma_0-\sigma_1)N_1(t)[1+o(1)],$$
$$\frac{y'(t)}{y(t)} = \frac{I'(t)R'(|\ln|\pi_\omega(t)||)}{(1-\sigma_0-\sigma_1)(1-\sigma_1)I(t)}\,[1+o(1)].$$

The next two theorems are devoted to the existence of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of equation (1). The first derivatives of such solutions are slowly varying functions as $t \uparrow \omega$, the fact creates difficulties in the investigation of such solutions.

Theorem 3. *For the existence of* $P_\omega(Y_0, Y_1, \pm\infty)$*-solutions of equation* (1) *the following conditions are necessary*

$$Y_0 = \begin{cases} \pm\infty, & if \ \ \omega = +\infty, \\ 0, & if \ \ \omega < +\infty, \end{cases} \qquad \pi_\omega(t)y_0^0 y_1^0 > 0 \ \ as \ \ t \in [a, \omega[\,.$$

If the function φ_0 *satisfies the condition* S *and*

$$\lim_{t\uparrow\omega} \frac{R'(|\ln|\pi_\omega(t)||)I_0(t)}{\pi_\omega(t)I_0'(t)} = 0, \tag{3}$$

then (3) *together with the following conditions are necessary and sufficient for the existence of* $P_\omega(Y_0, Y_1, \pm\infty)$*-solutions of equation* (1)

$$\lim_{t\uparrow\omega} y_1^0|I_0(t)|^{\frac{1}{1-\sigma_0-\sigma_1}} = Y_1, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)I_0'(t)}{I_0(t)} = 0, \quad y_1^0(1-\sigma_0-\sigma_1)I_0(t) > 0 \ \ as \ \ t \in [b_2, \omega[\,.$$

For such solutions the following asymptotic representations take place as $t \uparrow \omega$

$$\frac{y'(t)|y'(t)|^{-\sigma_0}}{\varphi_1(y'(t))\exp(R(|\ln|y(t)||))} = (1-\sigma_0-\sigma_1)I_0(t)[1+o(1)], \quad \frac{y'(t)}{y(t)} = \frac{1}{\pi_\omega(t)}\,[1+o(1)].$$

Theorem 4. *If in* (1) *the function* p *is a continuously differentiable, the function* φ_0 *satisfies the condition* S *and*

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)N'(t)}{R'(|\ln|\pi_\omega(t)||)N(t)}=0,$$

then with (3) *the following conditions are necessary and sufficient for the existence of* $P_\omega(Y_0,Y_1,\pm\infty)$*-solutions of equation* (1)

$$\lim_{t\uparrow\omega}y_1^0\exp\Big(\frac{1}{1-\sigma_0-\sigma_1}R(|\ln|\pi_\omega(t)||)\Big)=Y_1,\quad \alpha_0y_1^0(1-\sigma_0-\sigma_1)\ln|\pi_\omega(t)|>0\ \ as\ \ t\in[a,\omega[\,.$$

For such solutions the following asymptotic representations take place as $t\uparrow\omega$

$$\frac{|y'(t)|^{1-\sigma_0}}{\varphi_1(y'(t))\exp(R(|\ln|y(t)y'(t)||))}=\frac{|1-\sigma_0-\sigma_1|N(t)}{R'(|\ln|\pi_\omega(t)||)}\,[1+o(1)],\quad \frac{y'(t)}{y(t)}=\frac{1}{\pi_\omega(t)}\,[1+o(1)].$$

References

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