Construction of Partially Irregular Solutions of Linear Differential Systems in Critical Resonant Case

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Let D be a compact subset of $\mathbb{R}^n$ and $AP(\mathbb{R}\times D,\mathbb{R}^n)$ be a function space $f:\mathbb{R}\times D\to\mathbb{R}^n$. Each function $f(t,x)\in AP(\mathbb{R}\times D,\mathbb{R}^n)$ is continuous in the collection of variables and almost periodic in the t uniformly with respect to $x\in D$. According to [5, p. 60] we denote frequency modulus of function f as $\operatorname{Mod}(f)$, i.e. it is the smallest additive group of real numbers containing a set of Fourier exponents (frequencies) of function f. Throughout the paper we consider only systems written in the normal form. By the frequency modulus of a system of almost periodic equations we mean modulus of frequencies of its right-hand side. J. Kurzweil, O. Vejvoda in [6] showed that systems of ordinary almost periodic differential equations can have strongly irregular almost periodic solutions, i.e. intersection of frequency modulus this solutions with modulus of frequencies of system is trivially. Almost periodic solutions, frequency modulus of which contains only some frequencies of the system, were studied by A. K. Demenchuk in the articles [2–4] etc. This solutions are called partially irregular [4].

In this paper we investigate an existence problem for partially irregular almost periodic solutions of linear almost periodic system in the critical resonant case, where are purely imaginary eigenvalues with not simple elementary divisors of averaging of the coefficient matrix. The case of purely imaginary eigenvalues with simple elementary divisors was investigated in [4] and [1].

Let us consider the linear system

$$\frac{dx}{dt}=A(t)x+\varphi(t),\quad \operatorname{Mod}(A)\cap\operatorname{Mod}(\varphi)=\{0\},\ \ x\in\mathbb{R}^n, \tag{1}$$

and assume that $A(t)$ and $\varphi(t)$ are almost periodical such that intersection of frequency modules of the coefficient matrix $A(t)$ and driving forcing force $\varphi(t)$ is trivially. Almost periodic solutions $x(t)$, $\operatorname{Mod}(x)=\operatorname{Mod}(\varphi)$ of system (1) are called irregular forced [4]. Let us explore a existence problem of irregular with respect to $\operatorname{Mod}(A)$ almost periodic solutions $x(t)$ of system (1), i.e. such solutions that $(\operatorname{Mod}(x)+\operatorname{Mod}(\varphi))\cap\operatorname{Mod}(A)=\{0\}$, in critical resonant case.

Denote $\widehat{A}=\lim\limits_{T\to\infty}\frac{1}{T}\int\limits_0^T A(t)\,dt$, and $A_*(t)=A(t)-\widehat{A}$. Let Q_{A_*} be a constant nonsingular $n\times n$-matrix such that the first $n-d=s$ columns of matrix $A_*(t)Q_{A_*}$ are zero and the remaining columns are linearly independent. Let us consider the change of variables $x=Q_{A_*}y$, where $y=\operatorname{col}(y^{[s]},y_{[n-s]})$, $y^{[s]}=\operatorname{col}(y_1,\dots,y_s)$, $y_{[n-s]}=\operatorname{col}(y_{s+1},\dots,y_n)$. By $B^{[s,s]}$ and $B_{[n-s,s]}$ we denote respectively upper and lower blocks of $n\times s$-matrix that obtained from the matrix $B=Q_{A_*}^{-1}\widehat{A}Q_{A_*}$ deleting the last d columns (upper and lower indices indicate the dimensions of blocks). By $\psi(t)=Q_{A_*}^{-1}\varphi(t)$, $\psi(t)=\operatorname{col}(\psi^{[s]}(t),\psi_{[n-s]}(t))$ we denote the transformed driving force.

Lemma ([4]). *System* (1) *has almost periodic irregular solution $x(t)$ with respect to $Mod(A)$ if and only if:*

- *column rank of the matrix $A(t)-\widehat{A}(t)$ satisfies the inequality*
$$\operatorname{rank}_{\mathrm{col}} A_* = d < n; \tag{2}$$
- *the system*
$$\frac{dy^{[s]}}{dt} = B^{[s,s]}y^{[s]} + \psi^{[s]}(t) \tag{3}$$
has almost periodic solution $y^{[s]}(t)$ such that
$$\big(\mathrm{Mod}(y^{[s]}) + \mathrm{Mod}(\varphi)\big) \cap \mathrm{Mod}(A) = \{0\};$$
- *the following identity holds*
$$B_{[n-s,s]}y^{[s]}(t) + \psi_{[n-s]}(t) \equiv 0, \tag{4}$$
and $x(t) = Q_{A_*}\operatorname{col}(y^{[s]}(t), 0, \dots, 0)$.

Let $\alpha_k \pm i\beta_k$ ($k=1,\dots,k'$; $k'\leqslant n$; $i^2=-1$) be a eigenvalues of the matrix of coefficients $B^{[s,s]}$ of the reduced system (3). As noted above, in the article [1] a case of purely imaginary eigenvalues of matrix $B^{[s,s]}$ with simple elementary divisors was studied.

Suppose that there is a critical resonant case, when there is a pair of purely imaginary eigenvalues of matrix $B^{[s,s]}$ with multiplicity of two with not simple elementary divisors such that
$$\alpha_1=\alpha_2=0, \quad \beta_2=\beta_1\in Mod(\varphi), \quad \alpha_q\neq 0 \quad (q=3,\dots,k'). \tag{5}$$

Denote
$$G(t) = S_2^{-1}(t)\big(J_{B^{[s,s]}}S_2(t) - \dot{S}_2(t)\big), \quad S(t) = S_2^{-1}(t)S_1^{-1},$$
where
$$S_2(t) = \operatorname{diag}\big[e^{i\beta_1 t}, e^{i\beta_1 t}, e^{-i\beta_1 t}, e^{-i\beta_1 t}, 1, \dots, 1\big],$$
$\dot{S}_2(t)$ is a derivative of matrix $S_2(t)$, and matrix S_1 transforms matrix $B^{[s,s]}$ to the Jordan normal form, i.e.,
$$S_1^{-1}B^{[s,s]}S_1 = J_{B^{[s,s]}} = \operatorname{diag}\big[J_1, J_2, J_3, \dots, J_{k'}\big] = \operatorname{diag}\big[J_1, J_2, J\big],$$
$$J_1 = \begin{pmatrix} i\beta_1 & 1 \\ 0 & i\beta_1 \end{pmatrix}, \quad J_2 = \begin{pmatrix} -i\beta_1 & 1 \\ 0 & -i\beta_1 \end{pmatrix},$$
where J is a Jordan form, corresponding to the other eigenvalues of matrix $B^{[s,s]}$. Denote j-th row of matrix $g(t) = S(t)\psi^{[s]}(t)$ as $g_{(j)}(t)$ and j-th row of matrix $S(t)$ as $S_{(j)}(t)$.

Theorem. *Let coefficient matrix $A(t)$ and driving force $\varphi(t)$ of system* (1) *be almost periodic with trivial intersection of their frequency modules, and there be a critical resonant case* (5) *of the reduced system* (3). *Then:*

- *If system* (1) *has almost periodic irregular solution $x(t)$ with respect to* $\mathrm{Mod}(A)$, *then this solution is irregular forced, i.e.* $\mathrm{Mod}(x)\subseteq \mathrm{Mod}(\varphi)$.

- *System* (1) *has an irregular forced almost periodic solution if and only if condition* (2) *and the estimates*

$$\sup_t \bigg| \int_0^t S_{(2)}(\tau)\psi^{[s]}(\tau)\,d\tau \bigg| < \infty, \quad \sup_t \bigg| \int_0^t \bigg(\int_0^\tau S_{(2)}(\sigma)\psi^{[s]}(\sigma)\,d\sigma + S_{(1)}\psi^{[s]}(\tau) \bigg)\,d\tau \bigg| < \infty \tag{6}$$

hold and almost periodic solution $y^{[s]}(t)$ *of reduced system* (3) *satisfy the identity* (4)*.*

The lemma and the theorem allow us to find partially irregular solutions of linear differential systems. For example, consider the quasi-periodic differential system

$$\begin{aligned}
\frac{dx_1}{dt} &= -x_1 + x_4 + x_5,\\
\frac{dx_2}{dt} &= x_1 \sin\sqrt{5}\,t + \big(1+\sin\sqrt{5}\,t\big)x_2 - \big(1+\sin\sqrt{5}\,t\big)x_4,\\
\frac{dx_3}{dt} &= x_1 \cos\sqrt{5}\,t + x_2\cos\sqrt{5}\,t - x_4\cos\sqrt{5}\,t + x_5 + \cos t,\\
\frac{dx_4}{dt} &= -2x_1 + x_4 + x_5,\\
\frac{dx_5}{dt} &= -x_1\cos\sqrt{5}\,t - x_2\cos\sqrt{5}\,t - x_3 + x_4\cos\sqrt{5}\,t + \sin t,
\end{aligned} \tag{7}$$

wherein intersection of modules of frequencies of coefficients and driving force is trivially. System (7) has the solution

$$x = Q_{A*}y = \operatorname{col}\big(a\sin t - b\cos t, a\cos t + b\sin t, \sin t, (a+b)\sin t + (a-b)\cos t, 0\big), \tag{8}$$

where a, b are arbitrary real constants. The frequency of solution (8) coincide with the frequency of driving force and incommensurable with the frequency of the coefficients of system (7), therefore this solution is irregular forced.

References

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