Asymptotic Representations for Oscillatory Solutions of Higher Order Differential Equations

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We deal with an oscillation problem for the higher order nonlinear differential equation with a middle term

$$x^{(n)}(t)+q(t)x^{(n-2)}(t)+r(t)f(x(t))=0, \quad n\geq 3. \tag{0.1}$$

Precisely, we study the existence of oscillatory solutions of (0.1) which are bounded and not vanishing at infinity under the following assumptions:

(i) $q\in C^1[0,\infty)$, $q(t)\geq q_0>0$ for large t, and

$$\int\limits_0^\infty |q'(t)|\,dt<\infty.$$

(ii) $r\in C[0,\infty)$.

(iii) $f\in C(\mathbb{R})$ such that $f(u)u>0$ for $u\neq 0$.

Note that the function r may change its sign.

By a solution of (0.1) we mean a continuously differentiable function x up to n order defined on $[T_x,\infty)$, $T_x\geq 0$, such that satisfies (0.1) on $[T_x,\infty)$ and $\sup\{|x(t)|:\ t\geq T\}>0$ for $T\geq T_x$. As usual, a solution x of (0.1) is said to be *oscillatory* if there exists a sequence $\{t_n\}$ tending to infinity such that $x(t_n)=0$.

The assumption (i) assures that the second order linear equation

$$h''(t)+q(t)h(t)=0 \tag{0.2}$$

is oscillatory. Moreover, since q is bounded and has bounded variation on $[0,\infty)$, all solutions of (0.2) are bounded together with their derivatives.

In our approach equation (0.1) is studied as a perturbation of the linear differential equation

$$y^{(n)}(t)+q(t)y^{(n-2)}(t)=0. \tag{0.3}$$

From this point of view, our results are mainly motivated by the previous ones obtained by I. Kiguradze [5] for the special case $q(t)\equiv 1$, namely for the equation

$$x^{(n)}(t)+x^{(n-2)}(t)+r(t)f(x(t))=0. \tag{0.4}$$

It was shown in [5] that, if r is positive and sufficient large in some sense, then for n even every solution of (0.4) is oscillatory and for n odd every proper solution of (0.4) is oscillatory, or is vanishing at infinity together with its derivatives, or admits the asymptotic representation

$$x(t) = c(1 + \sin(t - \varphi)) + \varepsilon(t),$$

where c, φ are suitable constants and ε is a continuous function for $t \geq 0$ which vanishes at infinity. The existence of bounded oscillatory solutions for equations of type (0.1) has attracted the attention of many authors, see, e.g., the monograph [6], the papers [1–3] and references therein. Observe that if q is a positive constant, then (0.3) has oscillatory, bounded and not vanishing at infinity solutions. If q is not constant and (i) is satisfied, then, as already claimed, these properties remain to hold for the second order equation (0.2). Thus, it is natural to ask under which assumptions these properties are valid also for (0.3) and the more general case (0.1). Here, we give a positive answer to both these questions. In particular, our main results yield the existence of oscillatory solutions of (0.1), which are bounded and not vanishing at infinity. These results complete recent ones in [2] and extend similar ones in [5, Theorem 1.4], which are proved for equation (0.4). An application that concerns the influence of the perturbing term r on the change of the oscillatory character passing from (0.3) to the linear equation

$$x^{(n)}(t) + q(t)x^{(n-2)}(t) + r(t)x(t) = 0, \quad n \geq 3, \tag{0.5}$$

is given.

Below we use the following notation for the growth of unbounded solutions.

The symbol $g_1 = O(g_2)$ as $t \to \infty$ means, as usual, that there exists a constant M such that $|g_1(t)| \leq M|g_2(t)|$ for large t.

1 Oscillatory solutions in the linear case

Equations (0.2) and (0.3) are strictly related. When $q(t) \equiv 1$, a basis of the space of solutions of (0.3) is given by

$$t^j, \quad j = 0, 1, \dots, n-3, \quad \sin t, \quad \cos t. \tag{1.1}$$

In the general case, that is when q is not constant, it is easy to see that a basis of the space of solutions of (0.3) is given by

$$t^j, \quad j = 0, 1, \dots, n-3, \quad \Gamma_u, \quad \Gamma_v, \tag{1.2}$$

where

$$\Gamma_u = \int\limits_0^t (t-s)^{n-3} u(s)\, ds, \quad \Gamma_v = \int\limits_0^t (t-s)^{n-3} v(s)\, ds \tag{1.3}$$

and u, v are two independent solutions of (0.2).

The following existence result for oscillatory solutions of (0.2), which are bounded and not vanishing at infinity, holds.

Theorem 1.1 ([3, Theorem 2])**.** *Let $n \geq 3$, u be a nontrivial solution of* (0.2) *and*

$$\int\limits_0^\infty s^{n-3} |q'(s)|\, ds < \infty. \tag{1.4}$$

Then (0.3) *has an oscillatory solution* ϕ *such that*

$$\phi(t)=\begin{cases} u'(t)+\varepsilon(t) & \text{for } n \text{ odd},\\ u(t)+\varepsilon(t) & \text{for } n \text{ even}, \end{cases}$$

where ε *is a continuous function on* $[0,\infty)$ *and* $\lim\limits_{t\to\infty}\varepsilon(t)=0$. *In particular,*

$$0<\limsup_{t\to\infty}|\phi(t)|<\infty.$$

The following asymptotic expressions of the integrals in (1.3) is needed for proving Theorem 1.1.

Lemma 1.1 ([3, Lemma 5])**.** *Let* $n\geq 3$ *and* (1.4) *hold. If* u *is a nontrivial* (*oscillatory*) *solution of* (0.2), *then there exist constants* c_i, $i=0,1\dots,n-2$, $c_{n-2}\neq 0$, *and a function* ε *such that*

$$\Gamma_u(t)=\begin{cases} \sum\limits_{i=0}^{n-3}c_it^i+c_{n-2}u'(t)+\varepsilon(t), & \text{for } n \text{ odd},\\ \sum\limits_{i=0}^{n-3}c_it^i+c_{n-2}u(t)+\varepsilon(t), & \text{for } n \text{ even}, \end{cases}$$

where $\lim\limits_{t\to\infty}\varepsilon(t)=0$.

2 Oscillatory solutions in the nonlinear case

Let

$$F(u)=\max\big\{|f(v)|:\ -u\leq v\leq u\big\}.$$

The following criterion concerns the nonexistence of solutions of (0.1) vanishing at infinity.

Theorem 2.1 ([3, Theorem 1])**.** *Let* $n\geq 3$, $f\in C^1(\mathbb{R})$ *and*

$$\int\limits_0^\infty t^{n-3}|r(t)|\,dt<\infty. \tag{2.1}$$

Then (0.1) *does not have nontrivial solutions* x (*oscillatory or nonoscillatory*) *satisfying* $\lim\limits_{t\to\infty}x(t)=0$.

The following existence theorems hold.

Theorem 2.2 ([2, Theorem 1])**.** *Assume* $n\geq 3$. *Let for any positive constant* λ *and for some* $j=0,\dots,n-3$

$$\int\limits_0^\infty t^{n-3}F(\lambda t^j)|r(t)|\,dt<\infty.$$

Then for any solution y *of* (0.3) *such that* $y(t)=O(t^j)$ *as* $t\to\infty$, *there exists a solution* x *of* (0.1) *such that for large* t

$$x^{(i)}(t)=y^{(i)}(t)+\varepsilon_i(t),\ \ i=0,\dots,n-1,$$

where ε_i *are functions of bounded variation for large* t *and* $\lim\limits_{t\to\infty}\varepsilon_i(t)=0$, $i=0,\dots,n-1$.

Using Theorem 2.2 and Lemma 1.1 we get the asymptotic representations for solutions of (0.1).

Theorem 2.3 ([3, Theorem 4]). *Let $n \ge 3$ and u, v be two linearly independent solutions of* (0.2). *Assume* (1.4) *and for any positive constant λ*

$$\int_0^\infty t^{n-3}(\lambda t^{n-3})|r(t)|\,dt < \infty. \tag{2.2}$$

Then for any vector $(c_0, c_1, \dots, c_{n-1}) \in \mathbb{R}^n$ there exists a solution x of (0.1) *such that*

$$x(t) = \begin{cases} \displaystyle\sum_{i=0}^{n-3} c_i t^i + c_{n-2}u'(t) + c_{n-1}v'(t) + \varepsilon(t) & \text{for } n \text{ odd,} \\ \displaystyle\sum_{i=0}^{n-3} c_i t^i + c_{n-2}u(t) + c_{n-1}v(t) + \varepsilon(t) & \text{for } n \text{ even,} \end{cases} \tag{2.3}$$

where $\lim\limits_{t\to\infty} \varepsilon(t) = 0$. If, in addition, $f \in C^1(\mathbb{R})$ and there exists $M > 0$ such that

$$|f'(u)| \le MF(u) \ \text{ for large } |u|, \tag{2.4}$$

then the solution x given by (2.3) *is unique.*

Theorem 2.3 extends [5, Theorem 1.4] stated for (0.4) with $r(t) > 0$.

The argument for proving Theorems 2.2 and 2.3 is based on the Ascoli theorem and an iterative method, which can be also useful for a numerical estimation of solutions. Moreover, in [2] the cases $n = 3$ and $n = 4$ are studied in details.

As application, consider the Emden–Fowler type equation

$$x^{(n)}(t) + q(t)x^{(n-2)}(t) + r(t)|x(t)|^\lambda \operatorname{sgn} x(t) = 0, \quad \lambda > 0. \tag{2.5}$$

Then (2.4) is satisfied for any $\lambda > 0$ and (2.2) reads as

$$\int_0^\infty t^{(n-3)(\lambda+1)}|r(t)|\,dt < \infty.$$

Thus, according to Theorem 2.3, for a fixed vector $(c_0, c_1, \dots, c_{n-1})$ there exists a unique solution of (2.5) which has the asymptotic representation (2.3).

Another consequence of our results is the following.

Denote by S_y and S_x the solution space of (0.3) and (0.5), respectively. We say that (0.3) and (0.5) are *asymptotically equivalent*, if there exists a $1-1$ map $T : S_y \to S_x$ such that for every $y \in S_y$ there exists a unique $x \in S_x$ such that $T(y) = x$ and

$$\lim_{t\to\infty} (x(t) - y(t)) = 0.$$

Applying Theorems 2.1 and 2.2 we get the following.

Theorem 2.4 ([3, Theorem 5]). *Assume $n \ge 3$ and*

$$\int_0^\infty t^{2n-6}|r(t)|\,dt < \infty.$$

Then linear equations (0.3) *and* (0.5) *are asymptotically equivalent.*

The following example illustrates Theorem 2.3 and it is inspired from [4, page 113].

Example. Consider the equation

$$x^{(5)}(t) + q(t)x^{(3)}(t) + r(t)x^3(t) = 0. \tag{2.6}$$

where

$$q(t) = 1 + \Big(t + \frac{1}{2}\Big)^{-3} \sin t + \frac{2}{3}\Big(t + \frac{1}{2}\Big)^{-4} \cos t - \frac{1}{9}\Big(t + \frac{1}{2}\Big)^{-5} \cos^2 t$$

and $r \in C[0, \infty)$ and $t^8 r(t) \in L^1[0, \infty)$. A standard calculation shows that $q(t) > 1/2$ for large t and $q' \in L^1[0, \infty)$. Thus, assumption (i) is satisfied. Moreover, also (1.4) and (2.2) are verified. Since the function

$$u(t) = (\cos t)\bigg[\exp\bigg(8 \int\limits_0^t \frac{1}{(2s+1)^3} \cos s\, ds\bigg)\bigg]$$

is a solution of (0.2), see [4, page 113] with minor changes, in view of Theorem 2.3, for any vector $(c_0, \dots, c_3)$, equation (2.6) has the solution x given by

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 u'(t) + \varepsilon(t)$$

where $\lim\limits_{t\to\infty} \varepsilon(t) = 0$.

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