Functions Defined by n-tuples of the Lyapunov Exponents of Linear Differential Systems Continuously Depending on the Parameter Uniformly on the Semiaxis

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1 Introduction

For a given positive integer number n we denote by $\mathcal{M}^n$ the vector space of linear differential systems

$$\dot{x} = A(t)x, \;\; x \in \mathbb{R}^n, \;\; t \in \mathbb{R}^+ \equiv [0, +\infty), \tag{1.1}$$

with continuous and bounded on the semiaxis $\mathbb{R}^+$ matrix functions $A\colon \mathbb{R}^+ \to \operatorname{End}\mathbb{R}^+$ (we identify systems (1.1) with their coefficient matrices) with usual operations of addition and multiplying by real numbers. Let us introduce the two most commonly used in the theory of Lyapunov exponents topologies in the vector space $\mathcal{M}^n$: the *uniform* one given by the norm

$$\|A\| = \sup_{t \in \mathbb{R}^+} |A(t)|, \;\; A \in \mathcal{M}^n,$$

and the *compact-open* one given by the metric

$$\rho_C(A, B) = \sup_{t \in \mathbb{R}^+} \min\Big\{|A(t) - B(t)|, \frac{1}{t}\Big\}, \;\; A, B \in \mathcal{M}^n,$$

where $|A(t)| = \sup\limits_{|x|=1} |A(t)x|$. The resulting topological spaces we denote by $\mathcal{M}^n_U$ and $\mathcal{M}^n_C$, respectively.

The following definition of the Lyapunov exponents of system (1.1) is equivalent to the classical one [5, p. 34] and is more convenient for our purposes.

Definition 1.1. The Lyapunov exponents of system (1.1) are defined [2] by

$$\lambda_i(A) = \inf_{L \in G_i(S(A))} \sup_{x \in L \setminus \{0\}} \varlimsup_{t \to +\infty} \frac{1}{t} \ln |x(t)|, \;\; i = \overline{1, n},$$

where $S(A)$ is the vector space of solutions of system (1.1) and $G_i(V)$ is the set of i-dimensional subspaces of a vector space V.

In our notation the Lyapunov exponents are numbered in non-decreasing order, unlike [7].

Let M be a metric space. Consider a family

$$\dot{x} = A(t,\mu)x, \ \ x \in \mathbb{R}^n, \ \ t \in \mathbb{R}^+, \tag{1.2}$$

of linear differential systems depending on a parameter $\mu \in M$ and satisfying the property: for any fixed μ system (1.2) belongs to the space $\mathcal{M}^n$ (i.e. has a continuous and bounded on the semiaxis coefficient matrix). For any fixed $i \in \{1,\dots,n\}$ we put in correspondence with each $\mu \in M$ the i-th Lyapunov exponent of system (1.2) and as a result obtain the function $\Lambda_i^A \colon M \to \mathbb{R}$ called the i-th Lyapunov exponent of family (1.2). We identify families (1.2) with their coefficient matrices, the same as we do for systems (1.1).

Further we will consider families (1.2) with two different types of continuous dependence on a parameter $\mu \in M$. Matrix function A of family (1.2) represents a mapping $M \to \mathcal{M}^n$ defined by $\mu \mapsto A(\,\cdot\,,\mu)$. Therefore for families (1.2) definition of continuity in parameter depends on a topology in the space $\mathcal{M}^n$. Let $\mathcal{A}_C^n(M)$ denote the class of families (1.2) for which the mapping $M \to \mathcal{M}^n$ is continuous when $\mathcal{M}^n$ is endowed with the compact-open topology and let $\mathcal{A}_U^n(M)$ denote the class of families (1.2) for which the mapping $M \to \mathcal{M}^n$ is continuous when $\mathcal{M}^n$ is endowed with the uniform topology. In other words, the class $\mathcal{A}_C^n(M)$ consists of families (1.2) such that for any fixed $\mu \in M$ and any $T > 0$

$$\lim_{\mu\to\mu} \max_{t\in[0,T]} \big\|A(t,\nu) - A(t,\mu)\big\| = 0$$

holds, i.e. convergence is uniform on each line segment. The class $\mathcal{A}_U^n(M)$ consists of families (1.2) such that for any fixed $\mu \in M$

$$\lim_{\mu\to\mu} \big\|A(\,\cdot\,,\nu) - A(\,\cdot\,,\mu)\big\| = 0$$

holds, i.e. convergence is uniform on the whole semiaxis.

A natural problem stated by V. M. Millionshchikov [6] is to describe the Lyapunov exponents Λ_i^A of families (1.2) as functions on a metric space M. In a significant step towards its solution, V. M. Millionshchikov proved [6,8] that for each $i \in \{1,\dots,n\}$ and any $A \in \mathcal{A}_C^n$ the function Λ_i^A can be represented as the limit of a decreasing sequence of functions of the first Baire class. In particular, this means that Λ_i^A belongs to the second Baire class. Simple examples show that for families from $\mathcal{A}_C^n$ the Lyapunov exponents Λ_i^A, $i = \overline{1,n}$, can be everywhere discontinuous even starting from $n = 1$. M. I. Rakhimberdiev proved [10] that in the Millionshchikov theorem the number of Baire class cannot be reduced. An exact characterization of Lyapunov exponents of families from $\mathcal{A}_C^n$ is given in paper [4]: a family $A \in \mathcal{A}_C^n(M)$ satisfying the equality $\Lambda_i^A = f$ exists if and only if the function f is upper-limit (the definition is given below) and has an upper semicontinuous minorant. Moreover, in paper [4] the author proved that an n-tuple $(f_1, f_2, \dots, f_n)$ of functions $M \to \mathbb{R}^n$ coincides with the n-tuple $(\Lambda_1^A, \dots, \Lambda_n^A)$ of the Lyapunov exponents of some family $A \in \mathcal{A}_C^n(M)$ if and only if each function f_i satisfies conditions above and the inequalities $f_1(\mu) \le \dots \le f_n(\mu)$ hold for all $\mu \in M$.

It is easy to see that for any space M and family $A \in \mathcal{A}_U^1(M)$ (the only) Lyapunov exponent of family (1.2) is continuous. O. Perron gave [9] (see also [3, 1.4]) an example of a mapping $A \in \mathcal{A}_U^2([0,1])$ such that the largest Lyapunov exponent of family (1.2) is not upper semicontinuous. For any metric space M, positive integers n and $i \in \{1,\dots,n\}$ the full description of the i-th Lyapunov exponent of family $A \in \mathcal{A}_U^n(M)$ is given in paper [1]: a family $A \in \mathcal{A}_U^n(M)$ satisfying the equality $\Lambda_i^A = f$ exists if and only if the function f is upper-limit and has continuous minorant and majorant.

The main purpose of this report is to describe the set of n-tuples $\{(\Lambda_1^A, \dots, \Lambda_n^A) :\ A \in \mathcal{A}_U^n(M)\}$ of the Lyapunov exponents for any given metric space M and positive integer n.

Definition 1.2. We call a function $f\colon M \to \mathbb{R}$ upper-limit if there exists a sequence of continuous functions $f_k\colon M \to \mathbb{R}$, $k \in \mathbb{N}$, such that

$$f(\mu) = \overline{\lim_{k\to\infty}} f_k(\mu), \quad \mu \in M.$$

Remark 1.1. The property of a function $f\colon M \to \mathbb{R}$ being upper-limit is equivalent to each of the next conditions:

(1) the function f can be represented as the pointwise limit of a decreasing sequence of functions of the first Baire class;

(2) pre-image of every semi-interval $[r, +\infty)$, $r \in \mathbb{R}$, under the mapping f is a G_δ-set.

In the notation of the monograph [2, § 37.1] functions satisfying this condition constitute class $({}^*, G_\delta)$. The equivalence of conditions (1) and (2) is established in [2, § 37.1] and that of condition (2) and Definition 1.2 is demonstrated in [4, Remark 3].

2 Main result

Theorem. *Consider an arbitrary metric space M, an integer number $n \geq 2$ and a set of functions $f_i\colon M \to \mathbb{R}$, $i = \overline{1,n}$. A family $A \in \mathcal{A}^n_U(M)$ satisfying equalities $\Lambda^A_i = f_i$, $i = \overline{1,n}$, exists if and only if* (1) *the inequalities $f_1(\mu) \leq \cdots \leq f_n(\mu)$ hold for each $\mu \in M$ and* (2) *each function f_i, $i = \overline{1,n}$, is upper-limit and has continuous minorant and majorant. Moreover, if all functions f_i, $i = \overline{1,n}$, are bounded, then the coefficient matrix of family A can be chosen bounded.*

Remark 2.1. In the case $n = 1$ family (1.2) satisfying the required conditions exists if and only if the function f_1 is continuous.

Remark 2.2. It is easy to see that the conditions of the theorem above are stronger than those of an analogous theorem of paper [4]: our theorem requires the existence of continuous minorant and majorant for each of the given functions, while in [4] only the existence of an upper-semicontinuous minorant is required.

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