On Exponential Stability of Invariant Tori of a Class of Nonlinear Systems

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One of the important issues in the qualitative theory of multifrequency oscillations is the question of the stability of invariant sets of dynamical systems defined in the direct product of the m-dimensional torus and the n-dimensional Euclidean space. The main results were obtained in the works of A. M. Samoĭlenko [3]. In this paper, we have established new conditions for the exponential stability of a trivial torus of nonlinear extensions of a dynamical system on a torus, which are formulated in terms of the properties of the right-hand sides of the system not on the whole torus, but only on the set of non-wandering points. The obtained results are applied to the investigation of the stability of toroidal sets of one class of impulsive dynamical systems [4]. Relevant studies for linear extensions of dynamical systems on the torus were used in [1,5].

Consider the system of differential equations in the direct product m-dimensional torus $\mathcal{T}_m$ and n-dimensional Euclidean space $\mathbb{R}^n$

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = P(\varphi, x)x, \tag{1}$$

where $\varphi = (\varphi_1, \dots, \varphi_m)^T \in \mathcal{T}_m$, $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$, the function P is continuous on $\mathcal{T}_m \times \mathbb{R}^n$ and for all $x \in \mathbb{R}^n$ $P(\,\cdot\,, x), a(\,\cdot\,) \in C(\mathcal{T}_m)$, $C(\mathcal{T}_m)$ – continuous space 2π-periodic for each component φ_v, $v = 1, \dots, m$, functions defined on $\mathcal{T}_m$.

Let the following conditions be fulfilled:

$$\exists\, M > 0 \;\text{ that is }\; \forall\,(\varphi, x) \in \mathcal{T}_m \times \mathbb{R}^n \quad \|P(\varphi, x)\| \le M, \tag{2}$$

$$\begin{gathered} \forall\, r > 0 \;\; \exists\, L = L(r) > 0 \;\text{ that is }\; \forall\, x', x'' \;\; \|x'\| \le r, \;\; \|x''\| \le r, \\ \forall\, \varphi \in \mathcal{T}_m \quad \big\|P(\varphi, x'') - P(\varphi, x')\big\| \le L\|x'' - x'\|, \end{gathered} \tag{3}$$

$$\exists\, A > 0 \;\; \forall\, \varphi', \varphi'' \in \mathcal{T}_m \quad \big\|a(\varphi'') - a(\varphi')\big\| \le A\|\varphi'' - \varphi'\|. \tag{4}$$

The condition (4) guarantees that the system

$$\frac{d\varphi}{dt} = a(\varphi) \tag{5}$$

generates a dynamic system on $\mathcal{T}_m$, which we will mark by $\varphi_t(\varphi)$.

Definition ([2])**.** Point $\varphi \in \mathcal{T}_m$ is called the wandering point of a dynamic system (5) if there is a neighborhood $U(\varphi)$ and moment of time $T = T(\varphi) > 0$ such that

$$U(\varphi) \cap \varphi_t(U(\varphi)) = \varnothing \;\; \forall\, t \ge T.$$

Denote by Ω the set of nonwandering points of the dynamic system (5). Since $\mathcal{T}_m$ is a compact, then Ω is a non-empty, invariant, compact subset $\mathcal{T}_m$. In addition, the following lemma is true.

Lemma ([2])**.** *For all $\varepsilon > 0$ there exist $T(\varepsilon) > 0$, $N(\varepsilon) > 0$ such that for all $\varphi \notin \Omega$ the corresponding trajectory $\varphi_t(\varphi)$ is only a finite period of time which does not exceed $T(\varepsilon)$, outside ε-neighborhood set Ω, leaving this neighborhood set no more $N(\varepsilon)$ times.*

The main purpose of the work is to establish the exponential stability of the trivial torus $x = 0$, $\varphi \in \mathcal{T}_m$ of the system (1) in terms of the properties of the function $\varphi \mapsto P(\varphi, 0)$ on the set of non-lattice points Ω of a dynamical system (5), as well as apply the obtained results to the study of the stability of toroidal sets of impulse dynamical systems generated by the problem (1).

Let's denote $\varphi \in \mathcal{T}_m$, $x \in \mathbb{R}^n$

$$\widehat{P}(\varphi, x) = \frac{1}{2}\big(P(\varphi, x) + P^T(\varphi, x)\big),$$
$$\lambda(\varphi, x) \ \text{– biggest eigenvalue } \widehat{P}(\varphi, x).$$

Theorem 1. *Let the condition be fulfilled*

$$\forall\, \varphi \in \Omega \quad \lambda(\varphi, 0) < 0. \tag{6}$$

Then the trivial torus of system (1) *is exponentially stable, i.e. there are constants* $K > 0$, $\gamma > 0$, $\delta > 0$ *such as for all* $\varphi \in \mathcal{T}_m$ *as* $x^0 \in \mathbb{R}^n$, $\|x^0\| \le \delta$ *fair inequality*

$$\forall\, t \ge 0 \quad \|x(t, \varphi, x^0)\| \le K\|x^0\|e^{-\gamma t},$$

where $x(t, \varphi, x^0)$ *– solution to the Cauchy problem*

$$\frac{dx}{dt} = P(\varphi_t(\varphi), x)x, \quad x(0) = x^0.$$

Remark. From the proof of the theorem it follows that for arbitrary $\varphi \in \mathcal{T}_m$ the following inequalities are performed

$$\forall\, t \ge 0 \quad \exp\bigg\{\int_0^t \delta(\varphi_s(\varphi), r)\, ds\bigg\} \le Ke^{-\gamma t},$$

where

$$\delta(\varphi, r) = \max_{\|x\| \le r} \lambda(\varphi, x).$$

As an example, consider system (on $\mathcal{T}_1 \times \mathbb{R}^2$)

$$\frac{d\varphi}{dt} = -\sin^2\Big(\frac{\varphi}{2}\Big), \tag{7}$$

$$\begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{pmatrix} = \begin{pmatrix} -\cos(\varphi + x_1) & \sin(\varphi + x_2^2) \\ \sin(\varphi - x_2^3) & -\cos(\varphi + x_1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{8}$$

Dynamic system on $\mathcal{T}_1$, generated (7), has a set of nonwandering points

$$\Omega = \{\varphi = 0\}.$$

Symmetric matrix $P(0, \overline{0}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ has its own eigenvalues $\lambda_1 = \lambda_2 = -1$, so the condition (6) is fulfilled and by Theorem 1 trivial torus of system (7), (8) is exponentially stable.

As an application in the phase space $\mathcal{T}_m \times \mathbb{R}^n$ the impulsive system of differential equations is considered

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = P(\varphi, x)x, \tag{9}$$

$$\triangle x\big|_{\varphi \in \Gamma} = I(\varphi, x)x, \tag{10}$$

where are the functions a, P satisfie the conditions (2)–(5), I is continuous and limited to $\mathcal{T}_m \times \mathbb{R}^n$ and for all $x \in \mathbb{R}^n$ $I(\,\cdot\,,x) \in C(\mathcal{T}_m)$.

Impulse set Γ is given by equality

$$\Gamma = \big\{\varphi \in \mathcal{T}_m \mid \quad \Phi(\varphi) = 0\big\},$$

where $\Phi \in C(\mathcal{T}_m)$. Assume that $\forall\,\varphi \in \mathcal{T}_m$ there exist $\{t_i(\varphi)\}_{i=1}^{\infty} \subset (0,+\infty)$ – roots of an equation $\Phi(\varphi_t(\varphi)) = 0$, moreover,

$$\exists\,\theta > 0 \;\; \forall\,\varphi \in \mathcal{T}_m \;\; \forall\,i \geq 1 \quad t_{i+1}(\varphi) - t_i(\varphi) \geq \theta. \tag{11}$$

We will assume

$$\alpha = \max_{\varphi \in \Gamma} \|E + I(\varphi,0)\|,$$

where E – unit matrix.

Theorem 2. *Let the condition* (11) *be fulfilled and*

$$\forall\,\varphi \in \Omega \quad \frac{1}{\theta}\,\ln\alpha + \lambda(\varphi,0) < 0.$$

Then the trivial torus of system (8), (9) *is exponentially stable.*

References

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