On Existence, Uniqueness and Continuous Dependence from Initial Datum of Mild Solution for Neutral Stochastic Differential Equation of Reaction-Diffusion Type in Hilbert Space

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1 Introduction

The following initial-value problem is considered

$$d\biggl(u(t,x)+\int\limits_{\mathbb{R}^d} b(t,x,u(\alpha(t),\xi),\xi)\,d\xi\biggr)$$
$$=\bigl(\Delta_x u(t,x)+f(t,u(\alpha(t),x),x)\bigr)\,dt+\sigma\bigl(t,u(\alpha(t),x),x\bigr)\,dW(t,x),\ \ 0<t\le T,\ \ x\in\mathbb{R}^d, \tag{1}$$
$$u(t,x)=\phi(t,x),\ \ -r\le t\le 0,\ \ x\in\mathbb{R}^d,\ \ r>0, \tag{2}$$

where $\Delta_x \equiv \sum\limits_{i=1}^{d} \partial_{x_i}^2$ is d-measurable operator of Laplace, $\partial_{x_i}^2 \equiv \frac{\partial^2}{\partial x_i^2}$, $i\in\{1,\dots,d\}$, $W(t)=W(t,\,\cdot\,)$ is $L_2(\mathbb{R}^d)$-valued Q-Wiener process, $\{f,\sigma\}\colon [0,T]\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ and $b\colon [0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ are some given functions to be specified later, $\phi\colon [-r,0]\times\mathbb{R}^d\times\Omega\to\mathbb{R}$ is an initial-datum function and $\alpha\colon [0,T]\to[-r,\infty)$ is a delay-function.

Differential equations with delay have appeared as mathematical models of real processes, evolution of which depends on previous states. Number of works are devoted to investigation qualitative theory of stochastic differential equations with delay in finite-dimensional spaces. With regard to such equations in infinite-dimensional spaces, let us remark the work [3], where theorem on existence and uniqueness of mild solution to neutral stochastic differential equation in Hilbert space has been proved. But conditions of this theorem are formulated in an abstract form, therefore it is difficult to check them directly for concrete equations in specific spaces, e.g., for stochastic partial differential equations of reaction-diffusion type. For such equations abstract mappings are generated by real-valued functions as operator of Nemytskii. Thus our expectations to receive conditions in terms of coefficients of these equations, i.e. in terms of real-valued functions, are natural. If such conditions are found, it will be possible to check them easily while solving concrete applied problems. Equation (1), considered in our work, is special case of equation from the work [3]. It has an applied importance: it models behavior of various dynamical systems in physics and mathematical biology. Equations of such type are well known in literature and have a wide range of applications. The presence of an integral term in (1) turns this equation into nonlocal neutral stochastic equation of reaction-diffusion type.

2 Preliminaries

Throughout the article $L_2(\mathbb{R}^d)$ will denote real Hilbert space with an inner product $(f,g)_{L_2(\mathbb{R}^d)} = \int\limits_{\mathbb{R}^d} f(x)g(x)\,dx$ and the corresponding norm $\|f\|_{L_2(\mathbb{R}^d)} = \sqrt{\int\limits_{\mathbb{R}^d} f^2(x)\,dx}$. Let $\{e_n(x), n\in\{1,2,\dots\}\}$ be an orthonormal basis in $L_2(\mathbb{R}^d)$ such that $\sup\limits_{n\in\{1,2,\dots\}} \|e_n\|_{L_\infty(\mathbb{R}^d)} \le 1$. Let $(\Omega,\mathcal{F},\mathbf{P})$ be a complete probability space. We now define Q-Wiener $L_2(\mathbb{R}^d)$-valued process $W(t)=W(t,\,\cdot\,)$ as follows

$$W(t,x)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}\,e_n(x)\beta_n(t),\ \ t\ge 0,\ \ x\in\mathbb{R}^d, \tag{3}$$

where $\{\beta_n(t), n\in\{1,2,\dots\}\}\subset\mathbb{R}$ are independent standard one-dimensional Wiener processes on $t\ge 0$, $\{\lambda_n, n\in\{1,2,\dots\}\}$ is a sequence of positive numbers such that $\sum\limits_{n=1}^{\infty}\lambda_n<\infty$. Let $\{\mathcal{F}_t(dW), t\ge 0\}$ be normal filtration, generated by (3). It means that $\mathcal{F}_t(dW)$ is the least σ-algebra such that increments $W(t)-W(s)$ are measurable with respect to this σ-algebra for $0\le s\le t$. It is clear that $W(t)-W(s)$, $s\le t$, are independent from $\mathcal{F}_s(dW)$.

In what fellows, we will need some facts on the Cauchy problem for heat-equation

$$\begin{aligned}\partial_t u(t,x)&=\Delta_x u(t,x),\ \ t>0,\ \ x\in\mathbb{R}^d,\\ u(0,x)&=g(x),\ \ x\in\mathbb{R}^d.\end{aligned} \tag{4}$$

Let us denote

$$\mathscr{K}(t,x)=\begin{cases}\dfrac{1}{(4\pi t)^{\frac d2}}\exp\Big\{-\dfrac{|x|^2}{4t}\Big\}, & t>0,\\ 0, & t<0,\end{cases}\quad \text{– heat-kernel.}$$

Proposition 2.1 ([1, p. 47])**.** *If g in (4) belongs to $L_2(\mathbb{R}^d)$, then it's solution will be represented by the following formula*

$$u(t,x)=\int\limits_{\mathbb{R}^d}\mathscr{K}(t,x-\xi)g(\xi)\,d\xi,$$

an besides $u\in C^\infty((0,\infty)\times\mathbb{R}^d)$.

Proposition 2.2 ([1, pp. 242–244])**.** *Operators $S(t)\colon L_2(\mathbb{R}^d)\to L_2(\mathbb{R}^d)$, generating solution of the Cauchy problem (4) by the rule*

$$u(t,x)=(S(t)g(\,\cdot\,))(x)=\int\limits_{\mathbb{R}^d}\mathscr{K}(t,x-\xi)g(\xi)\,d\xi,$$

form an analytic contractive (C_0-)semi-group of operators, i.e. the following estimate is valid

$$\big\|(S(t)g(\,\cdot\,))(x)\big\|^2_{L_2(\mathbb{R}^d)}\le\|g(x)\|^2_{L_2(\mathbb{R}^d)},$$

and besides Laplacian Δ_x is an infinitesimal generator of this semi-group.

Proposition 2.3 ([2, p. 274])**.** *For partial derivatives of $\mathscr{K}$ the following estimate is true*

$$\big|\partial_t^r\partial_x^s\mathscr{K}(t,x)\big|\le c_{r,s}t^{-\frac d2-r-\frac s2}\exp\Big\{-\frac{c_0|x|^2}{t}\Big\},\ \ c_{r,s}>0,\ \ c_0<\frac14\,. \tag{5}$$

Proposition 2.4. *If g in (4) belongs to $L_1(\mathbb{R}^d)\cap L_2(\mathbb{R}^d)$, then solution of this problem will satisfy the following limit relations*

$$\lim_{|x|\to\infty} u(t,x)=0, \quad \lim_{|x|\to\infty} \partial_t u(t,x)=0. \tag{6}$$

◀ The proof follows from standard theorems on possibility to limit transition in Lebesgue integral and differentiability of integral by parameter via using estimate (5). ▶

From Propositions 2.1 and 2.4 we have the following result.

Proposition 2.5 ([2, p. 360])**.** *If relations (6) are valid, then for some $C_T>0$, depending only on T, solution of (4) will satisfy*

$$\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}(\Delta_x u(t,x))^2\,dx=\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\big\|D_x^2u(t,x)\big\|_d^2\,dx\le C_T\int\limits_{\mathbb{R}^d}\|D^2g(x)\|_d^2\,dx,$$

where $\nabla_x\equiv(\partial_{x_1}\cdots\partial_{x_d})^\top$, $D_x^2\equiv\begin{pmatrix}\partial_{x_1}^2 & \cdots & \partial_{x_1x_d}\\ \vdots & \ddots & \vdots\\ \partial_{x_dx_1} & \cdots & \partial_{x_d}^2\end{pmatrix}$ *is Hesse-operator,* $\|\cdot\|_d$ *is the corresponding norm of matrix.*

3 Formulation of the problem

The following assumptions are the main, assumed in the article.

3.1) $\alpha\colon[0,T]\to[-r,\infty)$ is function from $C^1([0,T])$ such that $0<\alpha'\le1$ (observe that there exist a constant $c>0$ and a unique point $0\le t^*\le T$ such that $\frac{1}{\alpha'}\le c$, $\alpha(t^*)=0$);

3.2) $\{f,\sigma\}\colon[0,T]\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$, $b\colon[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ are measurable with respect to all of their variables functions, and b is continuous by its first argument;

3.3) initial-datum function $\phi(t,\cdot,\omega)\colon[-r,0]\times\Omega\to L_2(\mathbb{R}^d)$ is $\mathcal{F}_0$-measurable random variable, independent from W, with almost surely continuous paths and such that

$$\begin{gathered}\mathbf{E}\phi^2(t)<\infty,\ \ -r\le t\le0,\\ \mathbf{E}\sup_{-r\le t\le0}\|\phi(t)\|^p_{L_2(\mathbb{R}^d)}<\infty,\ \ p>2;\end{gathered}$$

3.4) for $\{f,\sigma\}$, there exist a constant $L>0$ and a function $\chi\colon[0,T]\times\mathbb{R}^d\to[0,\infty)$ such that

$$\sup_{0\le t\le T}\int\limits_{\mathbb{R}^d}\chi^2(t,x)\,dx<\infty$$

and the following conditions of linear-growth and Lipschitz are valid

$$\begin{aligned}|f(t,u,x)|&\le\chi(t,x)+L|u|,\ \ 0\le t\le T,\ \ u\in\mathbb{R},\ \ x\in\mathbb{R}^d,\\ \big|f(t,u,x)-f(t,v,x)\big|&\le L|u-v|,\ \ 0\le t\le T,\ \ \{u,v\}\subset\mathbb{R},\ \ x\in\mathbb{R}^d,\\ |\sigma(t,u,x)|&\le L(1+|u|),\ \ 0\le t\le T,\ \ u\in\mathbb{R},\ \ x\in\mathbb{R}^d,\\ \big|\sigma(t,u,x)-\sigma(t,v,x)\big|&\le L|u-v|,\ \ 0\le t\le T,\ \ \{u,v\}\subset\mathbb{R},\ \ x\in\mathbb{R}^d;\end{aligned}$$

3.5) $|b(t,x,0,\xi)| \le b_1(x,\xi)$, $0 \le t \le T$, $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$, where function $b_1\colon \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ satisfies conditions

$$\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} b_1(x,\zeta)\,d\zeta\,dx < \infty, \quad \int\limits_{\mathbb{R}^d}\Big(\int\limits_{\mathbb{R}^d} b_1(x,\zeta)\,d\zeta\Big)^2\,dx < \infty;$$

3.6) there exists a function $l\colon \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ such that

$$\big|b(t,x,u,\xi) - b(t,x,v,\xi)\big| \le l(x,\xi)|u-v|, \ \ 0 \le t \le T, \ \ \{x,\xi\} \subset \mathbb{R}^d, \ \ \{u,v\} \subset \mathbb{R},$$

and l satisfies the following conditions

$$\int\limits_{\mathbb{R}^d}\sqrt{\int\limits_{\mathbb{R}^d} l^2(x,\zeta)\,d\zeta}\,dx < \infty, \quad \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} l^2(x,\zeta)\,d\zeta\,dx < \infty;$$

3.7) for each $x \in \mathbb{R}^d$, there exist partial derivatives $\partial_{x_i} b$, $\partial_{x_i x_j} b$, $\{i,j\} \subset \{1,\dots,d\}$, and for gradient-vector $\nabla_x b$ and Hesse-matrix $D_x^2 b$ the following condition of linear-growth by the third argument is true

$$|\nabla_x b(t,x,u,\xi)| + \|D_x^2 b(t,x,u,\xi)\|_d \le \psi(t,x,\xi)(1+|u|), \ \ 0 \le t \le T, \ \ \{x,\xi\} \subset \mathbb{R}^d, \ \ u \in \mathbb{R},$$

and for $D_x^2 b$ – Lipschitz condition

$$\big\|D_x^2 b(t,x,u,\xi) - D_x^2 b(t,x,v,\xi)\big\|_d \le \psi(t,x,\xi)|u-v|, \ \ 0 \le t \le T, \ \ \{x,\xi\} \subset \mathbb{R}^d, \ \ \{u,v\} \subset \mathbb{R},$$

where function $\psi\colon [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ is such that

$$\sup_{0 \le t \le T}\int\limits_{\mathbb{R}^d}\Big(\int\limits_{\mathbb{R}^d} \psi(t,x,\xi)\,d\xi\Big)^2\,dx < \infty, \quad \sup_{0 \le t \le T}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \psi^2(t,x,\xi)\,d\xi\,dx < \infty,$$

and besides for each point $x_0 \in \mathbb{R}^d$, there exist its vicinity $B_\delta(x_0)$ and a nonnegative function φ such that

$$\sup_{0 \le t \le T} \varphi(t,\,\cdot\,,x_0,\delta) \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d), \ \ \delta > 0,$$
$$\big|\psi(t,x,\xi) - \psi(t,x_0,\xi)\big| \le \varphi(t,\xi,x_0,\delta)|x-x_0|, \ \ 0 \le t \le T, \ \ |x-x_0| < \delta, \ \ \xi \in \mathbb{R}^d.$$

Definition 3.1. Continuous random process $u(t,\,\cdot\,,\omega) : [-r,T] \times \Omega \to L_2(\mathbb{R}^d)$ is called mild solution of (1), (2) if it

1) is $\mathcal{F}_t$-measurable for almost all $-r \le t \le T$;
2) satisfies the following integral equation

$$u(t,\,\cdot\,) = S(t)\Big(\phi(0,\,\cdot\,) + \int\limits_{\mathbb{R}^d} b\big(0,\,\cdot\,,\phi(-r,\zeta),\zeta\big)\,d\zeta\Big) - \int\limits_{\mathbb{R}^d} b\big(t,\,\cdot\,,u(\alpha(t),\xi),\xi\big)\,d\xi$$
$$-\int\limits_0^t \Delta_{(\,\cdot\,)}\Big(S(t-s)\int\limits_{\mathbb{R}^d} b\big(s,\,\cdot\,,u(\alpha(s),\zeta),\zeta\big)\,d\zeta\Big)\,ds$$

$$
\begin{aligned}
&+\int_0^t S(t-s) f\big(s, u(\alpha(s), \,\cdot\,), \,\cdot\,\big)\, ds \\
&+\int_0^t S(t-s) \sigma\big(s, u(\alpha(s), \,\cdot\,), \,\cdot\,\big)\, dW(s, \,\cdot\,), \ \ 0 \le t \le T, \\
u(t, \,\cdot\,) &= \phi(t, \,\cdot\,), \ \ -r \le t \le 0, \ \ r > 0.
\end{aligned}
$$

Remark 3.1. It is assumed in the definition above that all integrals make sense.

Our first result is concerned with existence and uniqueness of solution to (1), (2).

Theorem 3.1 (existence and uniqueness)**.** *Suppose that assumptions* 3.1–3.7 *are satisfied. Then, if*

$$
\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} l^2(x, \xi)\, d\xi\, dx < \frac{1}{4}\,,
$$

the Cauchy problem (1), (2) *has a unique for* $0 \le t \le T$ *mild solution.*

Remark 3.2. If we replace an initial range $[-r, 0]$ from (2) with $[s-r, s]$ for arbitrary $s \ge 0$, it will be possible to guarantee existence and uniqueness of mild solution to (1), (2) for $0 \le s \le t$.

Concerning continuation of mild solution to (1), (2) on the whole semi-axis $t \ge 0$, the following corollary is true.

Corollary 3.1. *If in Theorem* 3.1 *conditions* 3.4–3.7 *are valid for* $t \ge 0$*, then the Cauchy problem* (1), (2) *has a unique mild solution for* $t \ge 0$*.*

The next result is concerned with continuous dependence of u from the corresponding initial-datum function ϕ.

Theorem 3.2 (continuous dependence)**.** *Under the conditions of Theorem* 3.1*, there exists* $C(T) > 0$ *such that for arbitrary admissible initial-datum functions* ϕ *and* ϕ_1 *the following estimates hold*

$$
\mathbf{E} \sup_{0 \le t \le T} \big\| u(t, \phi) - u(t, \phi_1) \big\|^p_{L_2(\mathbb{R}^d)} \le C(T) \mathbf{E} \sup_{-r \le t \le 0} \| \phi(t) - \phi_1(t) \|^p_{L_2(\mathbb{R}^d)}, \ \ p > 2,
$$

where $u(t, \phi)$ *denotes solution* $u(t, x)$ *of* (1) *that satisfies* (2)*.*

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